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IMPACT OF MACRO SHOCKS ON SOVEREIGN DEFAULT PROBABILITIES

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DISCUSSION PAPER

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SINOPSE

Utilizamos modelos de macrofinanças para estudar a interação entre variáveis macro e a curva de juros soberana brasileira usando dados diários. Calculamos as probabilidades de *default* implícitas do modelo e uma medida do impacto de choques macro nas probabilidades. Uma extensão da estratégia de identificação de Dai e Singleton para modelos gaussianos com fatores latentes e observáveis foi descrita de modo a estimar nossos modelos. Entre as variáveis testadas, VIX é o fator macro mais importante afetando títulos e probabilidades de *default* de curto prazo, e a taxa curta do Federal Reserve (Fed) é o fator mais importante que afeta probabilidades de longo prazo.

ABSTRACT

We use macro finance models to study the interaction between macro variables and the Brazilian sovereign yield curve using daily data. We calculate the model implied default probabilities and a measure of the impact of macro shocks on the probabilities.

An extension of the Dai-Singleton identification strategy for Gaussian models with latent and observable factors is described in order to estimate our models.

Among the tested variables, VIX is the most important macro factor affecting short term bonds and default probabilities and the Fed short rate is the most important factor affecting the long term default probabilities.

SUMMARY

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Marco S. Matsumura*

November 30, 2006

Abstract

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1 Introduction

Credit risk is an important component of the yield curve of emerging countries. It is linked to some payment obligation and the possible failure of the obligor to honour it, thus affecting the required yield rate a government will face in order to finance itself. This measure will also be of great importance for emerging market firms, since the foreign financing will typically contain the country risk, so that firm borrowing rates are usually higher than sovereign rates. We are thus led to ask the following questions: What are the factors most affecting the sovereign yield rates? Which variables causes greater impact on default probabilities? We present an empirical investigation using affine term structure models with macro factors and default motivated by such questions.

There are two main lines of credit risk models, the structural and the reduced. In all models, the price of the defaultable bond will depend on the probability of default and on the expected recovery rate upon default. Giesecke (2004) provides a short introductory survey. Black and Scholes (1973) and Merton (1974) initiated the field by proposing the first structural models using option theory. Black and Cox (1976) introduced the basic structural model in which default occurs at the first time the process of the firm's assets crosses a given a default barrier. Many articles were built extending Black and Cox model. More recently, second generation models were introduced by Leland (1994) and Leland and Toft (1996) in which the firm's incentive structure is modeled to determine the default barrier endogenously, obtaining as a result its optimal capital structure. Default occurs when the structure of incentives suggests that it is optimal to the issuer to default or when the payment is impossible. This happens at the time the value of the shares falls to zero.

However, the cited articles treat the corporate credit risk case. The sovereign credit risk differs markedly from the corporate. Some reasons are listed bellow.

- A sovereign debt investor may not have recourse to a bankruptcy code at the default event.

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- Sovereign default can be a political decision. There exists a trade-off between the costs of making the payments and the costs of reputation, of having the assets abroad seized or of having access to international commerce impeded.
- The same bond can be renegotiated many times. Some contracts have cross-default or collective action clauses. Assets in the country cannot be used as a collateral.
- The government can opt for defaulting on internal or external debt.
- Also, one must take into account the role played by key variables such as exchange rates, fiscal dynamics, reserves in strong currency, level of exports and imports, GDP, inflation and many other macro variables.

Therefore, constructing a structural model for the case of a country is a more delicate question. It is not obvious how to model the incentive structure of a government and its optimal default decision, or what "assets" could be seized upon default. Moreover, post-default negotiation rounds regarding the recovery rate can be very complex and uncertain.

Not surprisingly, then, it is difficult to find structural model papers in the sovereign context. We opt for using reduced models, where the time of default is not directly modeled. It is a totally inaccessible stopping time which is triggered by the first jump of a given exogenous process with default intensity λ. A totally inaccessible stopping time is defined in the following. A predictable stopping time τ is one for which there exists a sequence of announcing stopping times $\tau_1 \leq \tau_2 \leq \dots$ such that $\tau_n < \tau$ and $\lim \tau_n = \tau$ for all $\omega \in \Omega$ with $\{\tau(\omega) > 0\}$. In the structural models, if the evolution of the assets follows a Brownian diffusion, then the time of default is a predictable stopping time. A stopping time is totally inaccessible if no predictable stopping time τ' can give any information about τ: $\mathbb{P}[\tau = \tau' < \infty] = 0$. Thus, in the case of the reduced model, the default always comes as a "surprise". This characteristic adds more realism to the modelling: the MinFins, Russian sovereign bonds, had a price drop of around 80% in the days immediately following the announcement of the default of the Russian domestic bond GKO in 1998.

Lando (1998) and Duffie and Singleton (1999) developed versions of reduced models in which the default risk appears as an additional instantaneous spread in the pricing equation. It has the advantage of being able to take an affine form. Duffie and Kan (1996) introduced the largely used family of affine models, offering a good compromise between flexibility and numerical tractability.

Duffie et al (2003) analyzes the case of the Russian bonds extending the reduced model to include the possibility of multiple defaults (or multiple "credit events", such as restructuring, renegotiation or change of regime). After estimating the model for the risk free reference curve on a first stage and then for defaultable Russian sovereign bonds on a second, they use model implied spreads to examine, for instance, what are the determinants of the spreads, what is the degree of integration between different Russian bonds and what is the correlation between the spreads and the macroeconomic series. Another paper applying reduced model to emerging markets is Pagès (2001). However, their models only use latent variables. Since macro factors are not explicitly inserted as state variables, they can only affect indirectly the latent factors. Also, the impact of changes of bond yields in macro factors cannot be measured inside the model. Duffie et al (2003) estimated the model in two steps: first the parameters relative to the FED yield curve, then those of the Russian yield curve.

Ang and Piazzesi (2003) were the first to estimate a term structure model with macro factors alongside latent factors in a discrete time affine model. They incorporate different Taylor rules into the short rate used in the no arbitrage pricing. In their model, the macro factors affect the entire yield curve. However, the interest rates do not affect the macro factors, which means the monetary policy is ineffective. They estimate in 2 steps: first the macro dynamics and then the latent dynamics conditional on the macro factors. Ang et al (2005) estimate another specification without this drawback using Monte Carlo Markov Chain. A Macro Finance literature has quickly emerged since their work (see Diebold et al 2005).

Amato and Luisi (2005) use a three-step procedure in a model with macro factors and default risk, where the reference curve, then the macro parameters and finally the spreads are estimated in a conditional way. The paper studies the corporate case. However, this has again the restrictive condition that the macro factors are not affected by the yield curve. Also, the conditioning the estimation in more than one step may lead to a sub-optimal solution.

Our model incorporates the advances brought by the above lines of research to study the impact of macro factors on a defaultable term structure. We provide the comparison among many trial models in the search of the macro factors most influencing the credit spreads and default probabilities. Also, using Ang and Piazzesi's approach, we can use impulse response and variance decomposition techniques to analyze the direct influence of observable macro factors on prices and default probabilities. In pure latent models, the unobservable factors are abstractions that can, at best, be interpreted as geometric factors summarizing the yield curve movements, as seen in Litterman and Scheinkman (1991).

However, before estimating the parameters, one must choose an identification strategy. Not all parameters of the multifactor affine model can be estimated, since there are linear operators on the parameter space that leave the short rate, and thus the yields, fixed. We propose an identification based on Dai and Singleton (2000) that exactly identifies the model. It is also used in Matsumura and Moreira (2006), which studies the Brazilian domestic market. Ang et al (2005) and Dai and Philippon (2004) present sub-identified models whose parameters can be arbitrarily rotated preserving the likelihood. Another article addressing identification of models with observable and latent factors is Pericoli and Taboga (2006). They propose an exact identification, but it is required that the mean reverting matrix Φ of the process driving the state factors have real and distinct eigenvalues.

We choose to use continuous-time modeling with high frequency Brazilian and US data because of the limitations on the size of historical series. When using Brazilian data, one must take into account that frequent changes of regime have occurred until recently, such as a change from fixed to floating exchange rate in a currency crisis in January 1999.

Our main model contain 5 state variables, one latent for the FED, one for an external macro factor, one for an internal macro factor, and two latent for the Brazilian sovereign yield curve. Macro variables tested are: 1) FED short rate, FED long rate, FED slope, VIX index of implied volatility of options on Standard & Poor index, exchange rate, Brazilian stock exchange Bovespa index, Brazilian future exchange interest rate swaps (short-term, long term, slope of the term).

Therefore, our objectives include: 1) analyzing the determinants of the term structure of the Brazilian sovereign interest rates; 2) measuring the forecasting performance of the models; 3) calculating default probabilities and measure the impact of macro shocks on them; 4) proposing an identification for affine models with macro factors.

We report that: 1) VIX and FED strongly affects the default probabilities in the short term and in the long term, respectively. 2) VIX has strong effect on Brazilian sovereign yields, more than any investigated domestic macro indicator. 3) Since the FED short rate affects more the default probabilities than the Brazilian domestic short rate, US monetary policy can cause more impact on the term structure of default probabilities than Brazilian monetary policy.

2 Model

2.1 Pricing

We derive the pricing equations in the affine model (Duffie and Kan, 1996) context. Fix the probability space $(\Omega, \mathbb{F}, P)$ and assume no arbitrage. The price at time t of a zero coupon bond paying 1 at the maturity date $t+\tau$ is

$$P(t,\tau) = E^{\mathbb{Q}}\left[\exp\left(-\int_t^{t+\tau} r_t dt\right) \mid \mathbb{F}_t\right]. \tag{1}$$

The conditional expectation is taken under the equivalent martingale measure $\mathbb{Q}$, $t+\tau$ is the maturity date, r_t is the stochastic instantaneous rate and $\mathbb{F}_t$ is the filtration at time t.

The state of the economy is represented by a vector $X_t \in \mathbb{R}^d$. In affine models, one assumes that the short rate is an affine function of the state vector:

$$r_t = \delta_0 + \delta_1 \cdot X_t. \tag{2}$$

The state vector X_t follows a Gaussian process with mean reversion, which is a particular case of an affine dynamics (Duffie et al, 2001), and a continuous-time equivalent of a Vector Autoregression (VAR). Under the objective $\mathbb{P}$-measure,

$$dX_t = K(\xi - X_t)dt + \Sigma dw_t. \tag{3}$$

The $d\times d$ and $d\times 1$ parameters K and θ represent the mean reversion coefficient and the long term mean short rate, and $\Sigma\Sigma^{\intercal}$ is the instantaneous variance-covariance matrix of the d-dimensional standard Brownian shocks w_t.

The risk premium is also affine in the state variables,

$$\lambda_t = \lambda_0 + \lambda_1 \cdot X_t. \tag{4}$$

Then, under the martingale measure $\mathbb{Q}$, using Girsanov, we have

$$dX_t = K^{\star}(\xi^{\star} - X_t)dt + \Sigma dw_t^{\star}, \tag{5}$$

where $dw_t^{\star} = dw_t + \lambda_t dt$, $w_t^{\star}$ being a standard $\mathbb{Q}$-Brownian motion, and

$$K^{\star} = K + \Sigma\lambda_1,\ \xi^{\star} = K^{\star -1}(K\xi - \Sigma\lambda_0). \tag{6}$$

The price of the bond can be found using multifactor Feyman-Kac. If

$$E^{\mathbb{Q}}\left[\exp\left(-\int_t^{t+\tau} r(X_u)du\right) \mid \mathbb{F}_t\right] = v(X_t, t, \tau), \tag{7}$$

then $v(x, t, \tau)$ must satisfy the following PDE:

$$\mathbb{D}v(x, t, \tau) - r(x)v(x, t, \tau) = 0, \tag{8}$$

$$v(x, t, 0) = 1,$$

where the operator $\mathbb{D}$ is given by

$$\mathbb{D}v(x, t, \tau) := v_t(x, t, \tau) + v_x(x, t, \tau) \cdot K^{\star}(\xi^{\star} - x) + \frac{1}{2}tr[\Sigma\Sigma^{\intercal} v_{xx}(x, t, v)]. \tag{9}$$

The solution is exponential affine on the state variables:

$$v(t, \tau, x) = e^{\alpha(\tau)+\beta(\tau)\cdot x}, \tag{10}$$

where

$$\beta\prime(\tau) = -\delta_1 - K^{\star\intercal}\beta(\tau) \tag{11}$$

and

$$\alpha\prime(\tau) = -\delta_0 + \xi^{\star\intercal}K^{\star\intercal}\beta(\tau) + \frac{1}{2}\beta(\tau)^{\intercal}\Sigma\Sigma^{\intercal}\beta(\tau). \tag{12}$$

Calculation of the explicit solution of the above system of ODE's is only possible in some special cases, such as when we have a diagonal K. However, Runge-Kuta numerical integration can solve equations (11) and (12) efficiently.

As a result, the yield is given by an affine function of the state variables, $Y(t,\tau) = -\frac{\alpha(\tau)}{\tau} - \frac{\beta(\tau)}{\tau} \cdot X_t$.

Define $A(\tau) = -\frac{\alpha(\tau)}{\tau}$ and $B(\tau) = -\frac{\beta(\tau)}{\tau}$, then

$$Y(t,\tau) = A(\tau) + B(\tau) \cdot X_t. \tag{13}$$

Stacking the equations for the K yield maturities we arrive at a more concise expression

$$Y_t = A + BX_t, \tag{14}$$

where $Y_t = (Y(t,\tau_1), ..., Y(t,\tau_K))^\intercal$. The factor loadings A and B will depend on the set of parameters

$$\Psi = (\delta_0, \delta_1, K, \theta, \lambda_0, \lambda_1, \Sigma), \tag{15}$$

that need to be estimated according to the data being used.

2.2 Likelihood

The log-likelihood is the *log* of the density function of the sequence of observed yields $(Y_{t_1}, ..., Y_{t_n})$. To calculate it we must first find the transition density of $X_{t_i}|X_{t_{i-1}}$, integrating the equation (3):

$$X_{t_i|t_{i-1}} = (1 - e^{-K(t_i - t_{i-1})})X_{t_{i-1}} + e^{-K(t_i - t_{i-1})}\theta + \int_{t_{i-1}}^{t_i} e^{-K(t_i-u)}\Sigma dw_u. \tag{16}$$

Using Ito's isometry formula, it follows that the stochastic integral term above is Gaussian with mean zero and variance

$$E\left[\int_{t_{i-1}}^{t_i} e^{-K(t_i-u)}\Sigma dw_u\right]^2 = \int_{t_{i-1}}^{t_i} e^{-K(t_i-u)}\Sigma\Sigma^\intercal (e^{-K(t_i-u)})^\intercal du. \tag{17}$$

This means that $X_{t_i|t_{i-1}} \sim N(\mu_i, \sigma_i^2)$, where

$$\mu_i = (1 - e^{-K(t_i - t_{i-1})})X_{t_{i-1}} + e^{-K(t_i - t_{i-1})}\theta, \tag{18}$$

and

$$\sigma_i^2 = \int_{t_{i-1}}^{t_i} e^{-K(t_i-u)}\Sigma\Sigma^\intercal (e^{-K(t_i-u)})^\intercal du. \tag{19}$$

Since $dt = t_i - t_{i-1}$is small, since daily frequency is used, a very good approximation to the integral (17) is

$$\sigma_i^2 \simeq e^{-Kdt}\Sigma\Sigma^\intercal (e^{-Kdt})^\intercal dt. \tag{20}$$

Thus,

$$X_{t_i|t_{i-1}} = \mu_i + \sigma_i \, N(0, \mathbb{I}), \tag{21}$$

with

$$\sigma_i \;= e^{-Kdt}\Sigma\sqrt{dt}. \tag{22}$$

Now suppose the vectors X_t and Y_t have the same dimension, that is, the number of yield maturities equals the number of state variables. Then, we can invert the linear equation (14) and find X_t as a function h of Y_t:

$$X_t = B^{-1}(Y_t - A) = h(Y_t). \tag{23}$$

Using change of variables, it follows that

$$\log f_Y(Y_{t_1}, ..., Y_{t_n}; \Psi) = \log f_X(X_{t_1}, ..., X_{t_n}); \Psi) + \log |\det \nabla h|^n \tag{24}$$

$$= \log \prod_{i=2}^{n} f_{X_{t_i}|X_{t_{i-1}}}(X_{t_i}; \Psi) + |\det \nabla h|^n \tag{25}$$

$$= \sum_{i=2}^{n} \log f_{X_{t_i}|X_{t_{i-1}}}(X_{t_i}; \Psi) + \log |\det \nabla h| \tag{26}$$

2.3 Chen-Scott

The procedure above restricts the number of yield maturities that can be used, because of the inversion (23) used to obtain the model implied state vector. If we want to use more data available, that becomes a problem, since the additional yields make the model singular. One solution is to follow Chen and Scott (1993), and add measurement errors to some yields. We choose this method in our continuous time versions. Let d and K be the number of state variables and of maturities. We select d maturities out of K to be priced without error. Let Y_t^1 represent the set of those yields at a given time. The other yields are denoted by Y_t^2, and they will have independent normal measurement errors $u(t,\tau) \sim N(0, \sigma_u^2(\tau))$.We have then

$$\begin{bmatrix} Y_t^1 \\ Y_t^2 \end{bmatrix} = \begin{bmatrix} A^1 \\ A^2 \end{bmatrix} + \begin{bmatrix} B^1 & 0 \\ B^2 & 1 \end{bmatrix} \begin{bmatrix} X_t \\ u_t \end{bmatrix}, \tag{27}$$

where $A = [A^1, A^2]^T$ and $B = [B^1, B^2]^T$ correspond to the maturities priced with and without errors, respectively. Call h the function that inverts the equation (27). The log likelihood becomes

$$\log f_Y(Y_{t_1}, ..., Y_{t_n}; \Psi) = \log f_X(X_{t_1}, ..., X_{t_n}); \Psi) + \log f_u(u_{t_1,...,}u_{t_n}) + \log |\det \nabla h|^n \tag{28}$$

$$= \log \prod_{t=2}^{n} f_{X_t|X_{t-1}}(h(X_t); \Psi) + \log \prod_{t=2}^{n} f_u(u_t) + |\det \nabla h|^n \tag{29}$$

$$= \sum_{t=2}^{n} (\log f_{X_t|X_{t-1}}(h(X_t); \Psi) + \log f_u(u_t)) + (n-1) \log |\det B^1|. \tag{30}$$

Alternatively, a Kalman Filter could be used, with all maturities having measurement errors.

2.4 Adding Default

An important component in the term structure of emerging countries is the spread due to the possibility of default of the bond. We use Duffie and Singleton's version of the reduced model of credit risk to study credit spreads. The price P^D of a defaultable bond is

$$P^D(t,\tau) = E^{\mathbb{Q}} \left[1_{[T>t+\tau]} \exp\left(-\int_t^{t+\tau} r_t dt \right) + W_T 1_{[T \le t+\tau]} \exp\left(-\int_t^{T} r_t dt \right) |\mathbb{F}_t \right]. \tag{31}$$

The first part is what the bond owner receives if the maturity time comes before the default time T, a stopping time. In case of default, the investor receives the random variable W_T at the default time. If T is doubly stochastic with intensity λ, if the recovery upon default is given by $W_T = (1 - l_T)P_{T^-}$, where $l(t)$ is the loss rate, and if other technical conditions are met, Lando (1998) and Duffie and Singleton (1999) prove that

$$P^D(t,\tau) = E^{\mathbb{Q}}\left[\exp(-\int_t^{t+\tau}(r_t + s_t)dt)|\mathbb{F}_t\right], \tag{32}$$

where $s_t = l_t\lambda_t$ is the spread due to the possibility of default.

We explain briefly the concept of doubly stochastic stopping time (see Schönbucher, 2003, Duffie, 2001). Define $N(t) = 1_{[T\leq t]}$ the associated counting process. It can be shown that $N(t)$ is a sub-martingale. Applying the Doob-Meyer theorem, we know there exists a predictable, nondecreasing process $A(t)$ called the compensator of $N(t)$. One property of the compensator is to give information about the probabilities of the jump time. The expected marginal increments of the compensator $dA(t)$ is equal to the probability of the default occurring in the next increment of time:

$$E\left[A(t+\Delta t) - A(t)|\mathbb{F}_t\right] = E\left[N(t+\Delta t) - N(t)|\mathbb{F}_t\right] \tag{33}$$

$$= P\left[N(t+\Delta t) - N(t) = 1|\mathbb{F}_t\right]. \tag{34}$$

An intensity process λ_t for $N(t)$ exists if it is progressively measurable and non negative, and if $A(t) = \int_0^t \lambda(s)ds$.It turns out, under regularity conditions, that

$$\lambda(t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}P[T \leq t + \Delta t|T > t]. \tag{35}$$

So, $\lambda(t)$ represents the evolution of the instantaneous probability of defaulting by $T + t$ if default has not occurred up to T.

Now, we derive the price a defaultable bond in our affine context. We assume that the instantaneous spread is an affine function of the state variables,

$$s_t = \delta_0^s + \delta_1^s \cdot X_t. \tag{36}$$

The state vector X_t is extended incorporating additional state variables relative to the defaultable yields, but continue to follow a mean reversion Gaussian process

$$dX_t = K(\theta - X_t)dt + \Sigma dw_t. \tag{37}$$

The discount rate is

$$R_t = r_t + s_t = \delta_0^r + \delta_0^s + (\delta_1^r + \delta_1^s)\cdot X_t = R(X_t). \tag{38}$$

If the risk premium process is affine too,

$$\lambda_t^s = \lambda_0^s + \lambda_1^s \cdot X_t, \tag{39}$$

the price of the defaultable bond will be exponential affine,

$$P^D(t,\tau) = \exp(\alpha^D(\tau) + \beta^D(\tau)\cdot X_t), \tag{40}$$

with α^d and β^d given by solutions of Riccati equations:

$$\frac{\beta^D(\tau)}{d\tau} = -(\delta_1^r + \delta_1^s) - K^{\star T}\beta^D(\tau), \tag{41}$$

$$\frac{\alpha^D(\tau)}{d\tau} = -(\delta_0^r + \delta_0^s) - K^{\star T}\theta^{\star T}\beta^D(\tau) + \frac{1}{2}\beta^D(\tau)^T\Sigma\Sigma^T\beta^D(\tau). \tag{42}$$

Thus, we have

$$Y^D(t,\tau) = A^D(\tau) + B^D(\tau) \cdot X(t), \tag{43}$$

where $A^D(\tau) = -\frac{\alpha^D(\tau)}{\tau}$, $B^D(\tau) = -\frac{\beta^D(\tau)}{\tau}$, or, piling the equations,

$$Y_t^D = A^D + B^D \cdot X_t. \tag{44}$$

The likelihood function turns out to be equal to the previous case, except by the increased dimension. Duffie at al (2003) opted to make a 2 step maximization in which the reference curve parameters are estimated first, following the estimation of the yield spread curve parameters conditional on the estimated parameters. They assumed a "triangular" form for the dynamics of the state variables. The American short rate is affected the Russian short rate, but not vice-versa. We use the same idea, observing that a one step procedure could be used (as is explained later), but would increase the computational complexity. Let $X_t = (X_t^1, X_t^1)$, where X_t^1 and X_t^2 denote the reference yield curve and the emerging market yield curve state vectors, respectively. For example, in the dynamics bellow,

$$\begin{bmatrix} dX_t^1 \\ dX_t^2 \end{bmatrix} = \begin{bmatrix} K^{11} & 0 \\ K^{21} & K^{22} \end{bmatrix} \left(\begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix} - \begin{bmatrix} X_t^1 \\ X_t^2 \end{bmatrix} \right) dt + \begin{bmatrix} \Sigma^{11} & 0 \\ \Sigma^{21} & \Sigma^{22} \end{bmatrix} dw_t, \tag{45}$$

$$\begin{bmatrix} \lambda_t^1 \\ \lambda_t^2 \end{bmatrix} = \begin{bmatrix} \lambda_0^1 \\ \lambda_0^2 \end{bmatrix} + \begin{bmatrix} \lambda_1^1 \\ \lambda_1^2 \end{bmatrix} \cdot \begin{bmatrix} X_t^1 \\ X_t^2 \end{bmatrix}, \tag{46}$$

the parameters $\Psi = (K^{11}, \theta^1, \lambda_0^1, \lambda_1^1, \Sigma^{11})$ are estimated in a first round, and then the other ones conditional on Ψ.

2.5 Adding Macro Factors

To incorporate macro factors, we extend our state vector of our economy to include observable macro variables M_t. Call the latent variables by θ_t. Then,

$$X_t = (M_t, \theta_t). \tag{47}$$

We follow the approach of Ang and Piazzesi (2002). The short rate will be a combination of a Taylor Rule and an affine model:

$$r_t = \delta_0 + \delta_{11} \cdot M_t + \delta_{12} \cdot \theta_t. \tag{48}$$

The above specification allows the possibility of studying the inter-relations between macroeconomic questions, such as monetary policy, and finance problems, such as derivative pricing. Also, the affine tractability is completely retained. The same calculations as before lead to

$$Y(t,\tau) = A(\tau) + B^o(\tau) \cdot M_t + B^u(\tau) \cdot \theta_t. \tag{49}$$

The likelihood function is calculated using again the Chen and Scott inversion procedure. Some maturities (Y_t^2) will have measurement errors u_t. Note that

$$\begin{bmatrix} M_t \\ Y_t^1 \\ Y_t^2 \end{bmatrix} = \begin{bmatrix} 0 \\ A^1 \\ A^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ B^{o\,1} & B^{u\,1} & 0 \\ B^{o\,2} & B^{u\,2} & 1 \end{bmatrix} \begin{bmatrix} M_t \\ \theta_t \\ u_t \end{bmatrix}. \tag{50}$$

Denote by h the function from the state (X_t, u_t) vector to (X_t^o, Y_t, Y_t^2). One obtains X_t^u inverting on Y_t^1:

$$X_t^u = (B^{u\,1})^{-1}(Y_t^1 - A^1 - B^{o\,1} \cdot M_t). \tag{51}$$

Then

$$\log f_Y(Y_{t_1},...,Y_{t_n};\Psi) = \log f_X(X_{t_1},...,X_{t_n});\Psi) + \log f_u(u_{t_1,...,}u_{t_n}) + \log|\det \nabla h|^n \tag{52}$$

$$= \log \prod_{t=2}^{n} f_{X_t|X_{t-1}}(X_t;\Psi) + \log \prod_{t=2}^{n} f_u(u_t) + |\det \nabla h|^n \tag{53}$$

$$= -(n-1)\log|\det B^{u\,1}| + \sum_{t=2}^{n} \log f_{X_t|X_{t-1}}(X_t;\Psi) + \log f_u(u_t), \tag{54}$$

as before.

In a model in which the macro factors are not affected by the yield curve like Ang and Piazzesi (2003), the parameters are also distributed in a triangular form such as in equation (45), so that the macro factors can be estimated separately in a first step.

Our estimations, like Ang et al (2005), allow macro factors and the yield factors to fully interact. Also, we use a two step estimation only when we want to determine the reference curve. We assume that the US yield curve is not affected by the Brazilian yield curve and that can be estimated in a first step. However, we show bellow how to implement a one-step estimation.

Observe that a model with macro factors can substitute a reduced form model if we look to the US yield curve as macro factors influencing the emerging curve. However, the interpretation of the spread as the instantaneous expected loss given by the Duffie and Singleton (1999) model will be lost, together with the calculation of model implied default probabilities of the next subsection.

2.6 Default Probabilities

The term structure of default probabilities is given by

$$\Pr(t,\tau) = E^{\mathbb{P}}\left[\exp\left(-\int_t^{t+\tau} s_t dt\right)|\mathbb{F}_t\right]. \tag{55}$$

Using the same technique as in the pricing case, one arrives at

$$\Pr(t,\tau) = \exp(\alpha^{\Pr}(\tau) + \beta^{\Pr}(\tau)\cdot X_t), \tag{56}$$

with $\alpha^{\Pr}$ and $\beta^{\Pr}$ given by solutions of Riccati equations:

$$\frac{\beta^{\Pr}(\tau)}{d\tau} = -\delta_1^s - K^T\beta^{\Pr}(\tau), \tag{57}$$

$$\frac{\alpha^{\Pr}(\tau)}{d\tau} = -\delta_0^s - K^T\theta^T\beta^{\Pr}(\tau) + \frac{1}{2}\beta^{\Pr}(\tau)^T\Sigma\Sigma^T\beta^{\Pr}(\tau). \tag{58}$$

Note that the objective measure is used. Thus, the log of the probabilities is again an affine function of the state variables,

$$\log \Pr(t,\tau) = \alpha^{\Pr}(\tau) + \beta^{\Pr}(\tau)\cdot X(t). \tag{59}$$

2.7 State Dependent Premium Parameters

We tested one version of the model in which the risk premia depended on a slowly varying macroeconomic fundamental, the Debt/GDP.

$$\lambda_t(F_t) = \lambda_0(F_t) + \lambda_1(F_t)\cdot X_t. \tag{60}$$

2.8 IRF and Variance Decomposition: continuous time version

Impulse response functions and variance decompositions are used to analyze the impact of macro shocks on yields and default probabilities. The time impulse response function in discrete time is

$$X_t = \Sigma\varepsilon_t + \Phi\Sigma\varepsilon_{t-1} + \Phi^2\Sigma\varepsilon_{t-2} + \Phi^3\Sigma\varepsilon_{t-3} + ... \tag{61}$$

Since $Y_t = A + BX_t$, the response of the yield curve to the shocks is

$$\begin{array}{cccccc} B\Sigma\varepsilon_t & B\Phi\Sigma\varepsilon_t & B\Phi^2\Sigma\varepsilon_t & B\Phi^3\Sigma\varepsilon_t & ... \\ t+0 & t+1 & t+2 & t+3 & ... \end{array}. \tag{62}$$

In continuous time, we have

$$X_{t_i|t_{i-k}} = e^{-K(t_i - t_{i-k})}X_{i-k} + \sum_{l=0}^{k-1}\int_{t_{i-k+l}}^{t_{i-k+l+1}} e^{-K(t_i-u)}\Sigma dw_u. \tag{63}$$

Using the approximation (17), it follows that the response of X_t to a shock ε_t in a interval of time of dt is

$$\begin{array}{ccccc} \Sigma\sqrt{dt}\varepsilon_t & e^{-Kdt}\Sigma\sqrt{dt}\varepsilon_t & e^{-2Kdt}\Sigma\sqrt{dt}\varepsilon_t & e^{-3Kdt}\Sigma\sqrt{dt}\varepsilon_t & ... \\ t+0 & t+1 & t+2 & t+3 & ... \end{array} \tag{64}$$

The response of the yield Y_t is

$$\begin{array}{ccccc} B\Sigma\sqrt{dt}\varepsilon_t & Be^{-Kdt}\Sigma\sqrt{dt}\varepsilon_t & Be^{-2Kdt}\Sigma\sqrt{dt}\varepsilon_t & Be^{-3Kdt}\Sigma\sqrt{dt}\varepsilon_t & ... \\ t+0 & t+1 & t+2 & t+3 & ... \end{array}, \tag{65}$$

and the response of the log of the survival probability $\log \Pr(t,\tau)$ is

$$\begin{array}{ccccc} \beta^{\Pr}\Sigma\sqrt{dt}\varepsilon_t & \beta^{\Pr}e^{-Kdt}\Sigma\sqrt{dt}\varepsilon_t & \beta^{\Pr}e^{-2Kdt}\Sigma\sqrt{dt}\varepsilon_t & \beta^{\Pr}e^{-3Kdt}\Sigma\sqrt{dt}\varepsilon_t & ... \\ t+0 & t+1 & t+2 & t+3 & ... \end{array}. \tag{66}$$

In discrete time, the Mean Squared Error of the s-periods ahead error $X_{t+s} - EX_{t+s|t}$ is

$$MSE = \Sigma\Sigma^\intercal + \Phi\Sigma\Sigma^\intercal\Phi^\intercal + \Phi^2\Sigma\Sigma^\intercal(\Phi^2)^\intercal + ... + \Phi^s\Sigma\Sigma^\intercal(\Phi^s)^\intercal. \tag{67}$$

The contribution of the j-th factor to the MSE of X_{t+s} will be then

$$\Sigma_j\Sigma_j^\intercal + \Phi\Sigma_j\Sigma_j^\intercal\Phi^\intercal + \Phi^2\Sigma_j\Sigma_j^\intercal(\Phi^2)^\intercal + ... + \Phi^s\Sigma_j\Sigma_j^\intercal(\Phi^s)^\intercal, \tag{68}$$

while the j-th factor contribution to the MSE of Y_{t+s} is

$$B\Sigma_j\Sigma_j^T B^\intercal + B\Phi\Sigma_j\Sigma_j^\intercal\Phi^\intercal B^\intercal + B\Phi^2\Sigma_j\Sigma_j^\intercal(\Phi^2)^\intercal B^\intercal + ... + B\Phi^s\Sigma_j\Sigma_j^\intercal(\Phi^s)^\intercal B^\intercal. \tag{69}$$

In continuous time, it turns out that the s-period ahead MSE of is the integral:

$$MSE = \int_t^{t+s} e^{-K(t+s-u)}\Sigma\Sigma^\intercal(e^{-K(t+s-u)})^\intercal dt \tag{70}$$

Hence, the contribution corresponding to the j-th factor in the variance decomposition of X_{t+s}, Y_{t+s} and $\log\Pr(t,\tau)$ at time t are

$$\begin{array}{c} \int_t^{t+s} e^{-K(t+s-u)}\Sigma_j\Sigma_j^\intercal(e^{-K(t+s-u)})^\intercal dt, \\ B^\intercal\left(\int_t^{t+s} e^{-K(t+s-u)}\Sigma_j\Sigma_j^\intercal(e^{-K(t+s-u)})^\intercal dt\right)B \\ (\beta^{\Pr})^\intercal\left(\int_t^{t+s} e^{-K(t+s-u)}\Sigma_j\Sigma_j^\intercal(e^{-K(t+s-u)})^\intercal dt\right)\beta^{\Pr}. \end{array} \tag{71}$$

3 Identification

The complete set of parameters are distributed as follows. The number of state variables is d, of yield maturities is m, and of latent variables is n.

$$\xi^{\intercal}, \xi^{*\intercal} \in \mathbb{R}^{d},\ \sigma_u^{\intercal} \in \mathbb{R}^{m-n}, \Sigma = \begin{pmatrix} \Sigma_{MM} & \Sigma_{M\theta} \\ \Sigma_{\theta M} & \Sigma_{\theta\theta} \end{pmatrix} \in \mathbb{R}^{d\times d} \tag{72}$$

$$K = \begin{pmatrix} K_{MM} & K_{M\theta} \\ K_{\theta M} & K_{\theta\theta} \end{pmatrix}, K^{*} = \begin{pmatrix} K^{*}_{MM} & K^{*}_{M\theta} \\ K^{*}_{\theta M} & K^{*}_{\theta\theta} \end{pmatrix} \in \mathbb{R}^{d\times d}$$

In order to identify the model, we extend the canonical identification of Dai and Singleton (2000) for the case with observable factors. In the appendix, we show that if we set $\xi^{\theta} = 0$, $\Sigma_{\theta\theta} = \mathbb{I}$ (the identity matrix), $\Sigma_{M\theta} = 0$, and impose that $K_{\theta\theta}$, $\Sigma_{\theta\theta}$ and Σ_{MM} are lower triangular, then the model is exactly identified. We subtract the sample mean from the macro factors, which allow us to make $\xi^{M} = 0$ and so $\xi = 0$. It can be shown that Ang et al (2005) and Dai and Philippon (2004) models are not fully identified. On the other hand, others authors such as Hördahl et al (2004) and Amato and Luisi (2005) use over-identifying restrictions that are not motivated by economic reasons. Pericoli and Taboga (2006) correctly points out the way to achieve an exact identification. However, their identification requires that K have real and distinct eigenvalues, whereas ours do not make any assumption. Actually, there are many more identifications, and it is not clear how to choose among the possibilities. Matsumura and Moreira (2006) use the same identification proposed here to study the Brazilian domestic term structure.

Invariant transformations on the parameter space can change the impulse response functions of the latent factors arbitrarily. And if the model is sub-identified, many sets of parameters can be found for the same specification and data. Figures 16, 17 and 18 show an actual example of a model estimated with one latent and one macro factor, using FED data and VIX as the macro factor. The invariant transformation has two degrees of freedom. We estimated thrice the same specification with only one degree of freedom fixed, each time from a different starting vector, and each maximization was carried out successfully and arrived at the same maximal value. Then, we calculated the corresponding impulse response function. It is clear from the pictures that the response of the FED latent factor to a VIX impulse is varies arbitrarily.

However, when we are interested in models properties with respect to observable factors, the choice of the specification does not matter. It can be shown that the invariant transformations preserve the likelihood and the impulse response functions of the yield. The impact of a macro shock on the yield curve should be the same for all specifications, leaving aside numerical issues.

We assume that the state factors have a given intertemporal causality ordering in Σ as in the VAR literature. The FED rate is always the more exogenous factor, followed by the VIX, the domestic macro factors and the latent factors.

We did impose a slight super identifying restriction because our $K^{*}_{\theta\theta}$ is also lower triangular.

Summing up, we have:

$$\Sigma = \begin{pmatrix} \Sigma_{MM} & 0 \\ 0 & \mathbb{I} \end{pmatrix},\ \xi = 0,\ \text{and } K_{\theta\theta}, K^{*}_{\theta\theta}, \Sigma_{MM} \text{ lower triangular.} \tag{73}$$

A especial case is obtained when we set $K_{M\theta} = K^{*}_{M\theta} = 0$, which is called *macro-to-yield*, since the macro factors affect but are not affected by the financial latent factors. Another case is the *yield-to-macro*, in which $K_{\theta M} = K^{*}_{\theta M} = 0$. Here, yield curve affect macro factors but not vice-versa in the transition equation $dX_t = K(\xi - X_t)dt + \Sigma dw_t$. However, macro factors still affect the yield curve through the short rate equation, $r_t = \delta_0 + \delta_1 \cdot X_t$. The two restricted specifications are called *unilateral*, while the unrestricted is called *bilateral*.

We estimated 3 families of increasing difficulty specifications: A) Macro-to-yield models with one macro factor and two latent factors for the Brazilian yield curve; B) Macro-to-yield models with one macro factor, one FED latent factor, and two Brazilian latent factors; and C) Bilateral models with two macro factors, one FED and two Brazilian latent factors.

4 Estimation

The parameters are found maximizing the log-likelihood with respect to the parameters given the series of yields. The maximum likelihood estimation produces asymptotically consistent, non-biased and normally distributed estimators. Let $L = \log f_Y$. When $T \to \infty, \hat{\psi} \to \psi$ a.s.,

$$T^{\frac{1}{2}}(\hat{\psi} - \psi) \to N(0, \Omega) \text{ in distribution,} \tag{74}$$

where

$$\Omega^{-1} = E\left(\frac{\partial L(Y;\psi)}{\partial \psi}\frac{\partial L(Y;\psi)^T}{\partial \psi}\right) = -E\left(\frac{\partial^2 L(Y;\psi)}{\partial \psi^2}\right) \tag{75}$$

using the information inequality. Note that to calculate Ω we would have to know the true parameters. An estimator for Ω^{-1} is the empirical Hessian

$$\hat{\Omega}^{-1} := -\frac{1}{n}\sum_{t=1}^{n}\left(\frac{\partial^2 L_t(Y;\hat{\psi})}{\partial \psi^2}\right),$$

where L_t represents the likelihood of the vector with t elements. More details can be found in Davidson and Mackinnon (1993, Chapter 8).

We calculate the confidence intervals for the parameter estimations using equation the empirical Hessian and the Central Limit Theorem. If the number of observations n is large enough, then the variance of $\hat{\psi} - \psi$ will be given by the diagonal of $N(0, \Omega/n)$.Alternatively, one could obtain the confidence interval via simulation.

Our estimation strategy consisted in may trial optimizations using Matlab. To deal with problem of multiple local maxima, we started with the simpler macro-to-yield models with less parameters, choosing different starting points in the numerical optimization. The models with higher dimensions used parameters of sub-models as starting points when possible. For example, the FED 1D + VIX + BR 2D model used some parameters of the VIX + BR 2D model. Then, new trials from other random vectors were conducted and compared. The maximal result was then chosen. Although this procedure may be path-dependent, the "curse of dimensionality" does not allow one to use a grid of random starting points.

4.1 Data

We use the constant maturity zero-coupon term structure from BM&F (the Brazilian Futures exchange) interest rate swaps, the FED constant maturity zero-coupon yield curve, the constant maturity zero-coupon term structure of spreads from Bloomberg, the Chicago Board Options Exchange Volatility Index - VIX -, created from S&P 500 index options implied volatilities, the Brazilian Real/US dollar exchange rate and Bovespa index of the Brazilian Stock Exchange most traded firms, and finally the Brazilian Government Debt over GDP.

The sample used for the estimation begins on February 17^{th}, 1999, and ends on September 15^{th}, 2004, comprising 1320 days. More 200 days of available data, finishing on July 21^{st}, 2005, were

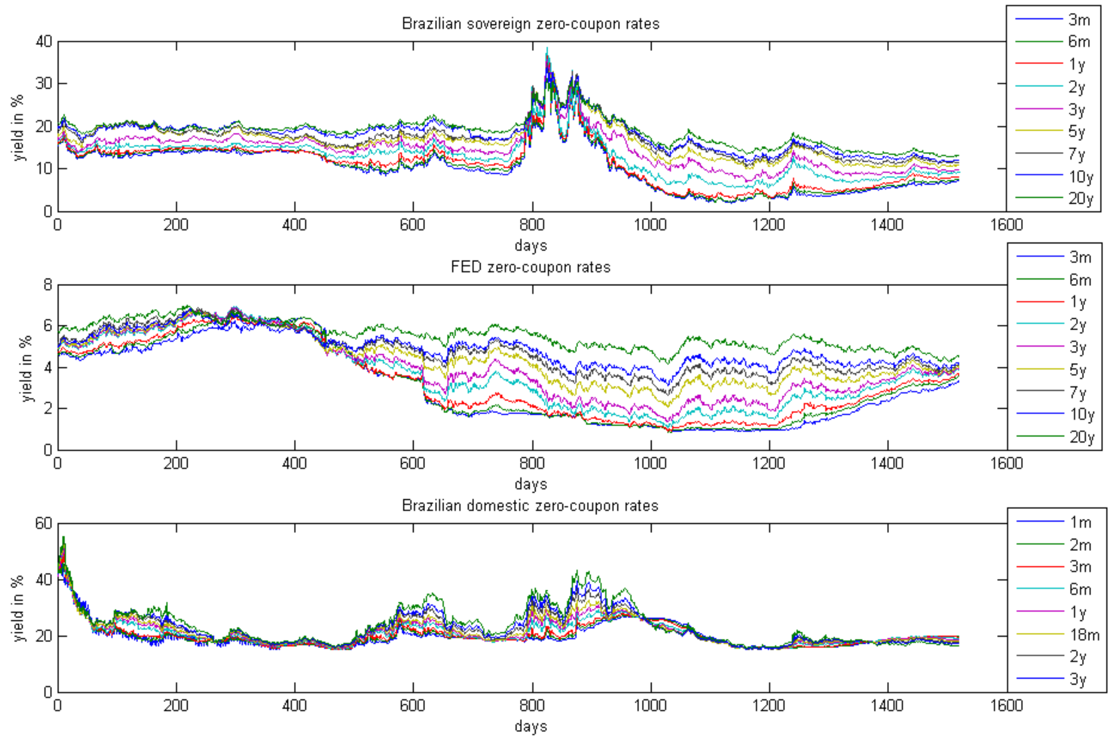

Figure 1: Brazilian domestic and sovereign zero coupon yield curve, and FED zero coupon term structure.

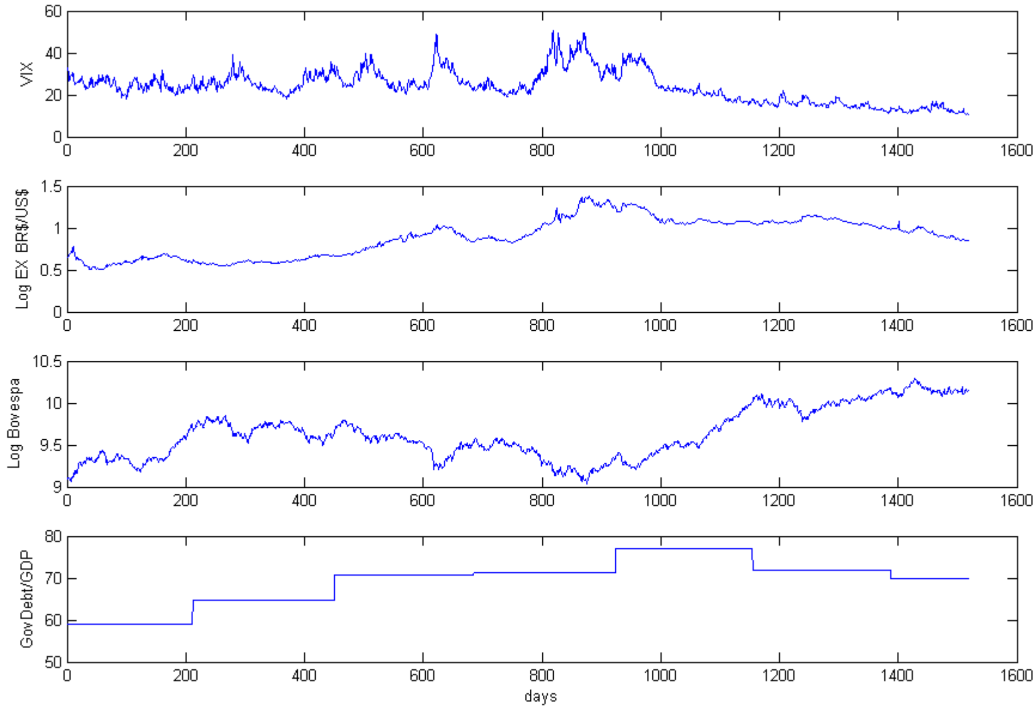

Figure 2: The option implied volatility index VIX, the log of the Brazilian RS$/US$ exchange rate, the log of the Bovespa Stock Exchange Index, and the Government Debt over the GDP.

separated for testing the forecasting performance. The maturities used in the FED and the sovereign Brazilian yield curve are the same: {3m, 6m, 1y, 2y, 3y, 5y, 10y, 20y}. We choose 3m and the 5y as the yield maturities priced without without measurement errors in the Chen-Scott inversion. We took the log of the exchange rate and of the Bovespa index, since our model is linear on the state variables. The Debt series have yearly frequency, and was used only in the model with variable premium parameters.

5 Results

5.1 Macro-to-yield without default

We begin presenting and comparing the simplest specification, whose main utility will be to the select macro factors to be used as state variables in the other models. The trial models have 3 state variables, $X = (M, \theta_1, \theta_2)$, one macro and two latent, characterized by the *macro-to-yield* dynamics. The following macro variables are used: 1) VIX, 2) BR Real/US Dollar exchange rate, 3) Bovespa, 4) BM&F 1-month yield, 5) BM&F 3-years yield and 6) BM&F slope = 3y - 1m yields, 7) FED 1-month yield, 8) FED 10-years yield and 9) FED slope=10y - 1m yields. We present in Table 1 A) the log-likelihood divided by the number of observations, B) the number of parameters, C) the in-sample adjustment and D) the out-of-sample forecasting performance, or Theil-U, E) the correlation between the latent factors and the sovereign level and slope, and F) the mean of the measurement errors in basis points.

The definitions of the adjustment and Theil-U are: the standard deviation, of a 1-month forecasting error of selected maturities, normalized by the standard deviation of a model that follows a random walk, measured in-sample or out-of-sample, respectively.:

$$\text{Adjustment, Theil-U} = \left(\frac{\sum_t (Y_t - \widehat{Y}_{t|t-21})^2}{\sum_t (Y_t - Y_{t-21})^2} \right)^{\frac{1}{2}}. \tag{76}$$

The out-of-sample forecasting performance was measured using the last 200 days of the sample.

Bellow, the results of the Macro 1D unilateral models are shown.

Table 1.
Summary of Macro 1D unilateral + BR 2D models.

	VIX	EX	Bove	bmf1m	bmf3y	bmfsl	fed1m	fed10y	fedsl
LL/T	44.66	44.25	44.25	44.75	44.79	44.95	47.52	47.46	47.07
#Par	26	26	26	26	26	26	26	26	26
M(In)	1.14	1.04	1.10	1.03	1.03	1.09	1.04	1.03	1.06
TU3m	2.02	0.76	2.58	2.51	2.50	6.00	2.16	2.15	2.21
TU1y	1.54	1.27	2.77	2.48	2.41	4.05	2.32	2.71	2.26
TU5y	1.37	0.96	2.03	1.33	1.31	3.98	1.27	1.75	1.05
TU10y	2.85	3.73	1.39	2.86	2.88	5.24	2.13	2.37	2.65
c(θ_1,s)	−0.20	−0.37	−0.29	−0.59	−0.56	−0.57	−0.66	−0.69	−0.61
c(θ_1,l)	−0.39	−0.18	−0.30	0.07	0.02	0.04	0.17	0.21	0.10
c(θ_2,s)	−0.80	−0.90	−0.89	−0.58	−0.59	−0.83	−0.58	−0.40	−0.59
c(θ_2,l)	0.99	0.83	0.98	0.94	0.94	1.00	0.94	0.84	0.94
M(σ_u)	72	88	74	82	83	77	82	81	83

The results show that:

1. The in-sample adherence and the out-of-sample forecasting performance of the specifications are similar.

2. There is a good fitting of the model to the data, with a mean error around 80 basis points.

3. The monetary factor θ_2 is highly correlated to the level of the yield curve. Except for the first 3, θ_1 is highly correlated to the slope.

4. The higher likelihood of the FED specifications can be attributed to better adjustment of FED in the transition equation $X_t|X_{t-1}$, or to the greater information brought by the FED rates.

5. The models show low forecasting performance for the majority of the maturities.

Next table measures the proportion the macro factors explain in the variance decompositions for forecast horizons of {1m, 9m, 18m) ahead for the {3m, 3y, 20y}-yields.

Table 2.
Variance decomposition of yields. Macro 1D unilateral + BR2D models.

Hor	Yi	VIX	EX	Bove	bmf1m	bmf3y	bmfsl	fed1m	fed10y	fedsl
1m	3m	0.15	0.07	0.01	0.00	0.00	0.16	0.00	0.00	0.00
1m	3y	0.23	0.09	0.00	0.00	0.00	0.23	0.00	0.04	0.00
1m	20y	0.54	0.09	0.06	0.00	0.00	0.50	0.00	0.08	0.00
9m	3m	0.31	0.07	0.22	0.00	0.00	0.46	0.00	0.02	0.06
9m	3y	0.46	0.11	0.13	0.00	0.00	0.61	0.00	0.10	0.07
9m	20y	0.69	0.14	0.21	0.00	0.00	0.79	0.00	0.16	0.07
18m	3m	0.31	0.09	0.33	0.00	0.00	0.47	0.00	0.06	0.16
18m	3y	0.46	0.12	0.21	0.00	0.00	0.61	0.00	0.16	0.17
18m	20y	0.69	0.16	0.27	0.00	0.00	0.79	0.00	0.22	0.17

Table 2 has the purpose of comparing the importance of the different macro variables for the sovereign yield curve. It will be used as a criterion to select those to be used in the later models. The ordering of the impact is the following:

- Greater effect: VIX and BM&F slope.
- Some effect: exchange rate, 10 years FED yield, FED slope, Bovespa index.
- Negligible effect: BM&F 1 month and 3 years yield, FED 1 month yield.

5.2 Macro-to-yield with default

Now, default risk is introduced into the previous specification by adding the FED latent factor besides the macro factor and the two Brazilian latent factors. Using the constant maturity zero-coupon US term structure, the parameters corresponding to the FED latent factor together with the model implied latent factor are estimated in a first step. In the second step, the other parameters are estimated conditional on the first step.

Table 3.
Summary of FED 1D + Macro {0 or 1}-D unilateral + BR 2D.

—	—	fund	VIX	bmf sl	bmf 3y	fed sl
LL/T	42.14	42.14	46.97	48.16	47.76	49.86
#Par	22	24	33	35	35	33
M(In)	1.05	1.05	1.14	1.01	1.02	1.03
TU3m	1.72	1.74	1.30	2.46	0.72	8.58
TU1y	1.45	1.45	1.58	1.77	1.70	6.25
TU5y	1.99	2.01	1.27	1.01	1.24	3.57
TU10y	5.68	5.79	3.30	3.74	1.92	1.44
c(θ_1,s)	−0.28	−0.39	−0.37	−0.42	−0.19	0.17
c(θ_1,l)	−0.28	−0.18	−0.23	−0.15	−0.36	−0.24
c(θ_2,s)	−0.74	−0.68	−0.68	−0.71	−0.96	−0.71
c(θ_2,l)	0.99	0.98	0.97	0.98	0.86	0.96
M(σ_u)	90	90	71	66	72	79

In Table 3, we show a summary of results of the model. It shows results similar to the Table 1. The first column show a model without macro factors. The second column corresponds to a model similar to the first one, but in which the risk premium parameters are functions of a slowly varying macroeconomic fundamental, the Debt/GDP. The others are macro-to-yield models. Bellow, we present the variance decomposition of the {3m, 3y, 20y}-yields of those models.

The inclusion of the FED produces a gain of likelihood and of fitting. There is still low forecasting performance and the θ_2 remains highly linked to the mean level of the yields.

The forth table details the contribution of each state factor. The first column identifies the time span between the shock and the response. The second column identifies the factor that is originating the impulse. How much it will impact on the selected yields will depend on the specification, and is listed from the forth column to the last.

Table 4.
Variance decomposition of yields. Fed 1D + Macro {0,1}-D unilateral + BR 2D models. Horizons of 1m, 9m and 18m ahead.

Horiz	Imp	Resp	–	fund	VIX	bmf sl	bmf 3y	fed sl
1m	FED	B3m	0.00	0.00	0.00	0.00	0.01	0.00
		B3y	0.00	0.00	0.00	0.00	0.01	0.00
		B20y	0.00	0.00	0.00	0.00	0.01	0.00
	Macr	B3m			0.15	0.01	0.01	0.02
		B3y			0.25	0.02	0.00	0.01
		B20y			0.56	0.05	0.04	0.01
	θ_1	B3m	0.13	0.14	0.10	0.25	0.30	0.51
		B3y	0.01	0.01	0.01	0.08	0.79	0.24
		B20y	0.18	0.17	0.09	0.07	0.85	0.02
	θ_2	B3m	0.87	0.86	0.75	0.74	0.68	0.47
		B3y	0.99	0.99	0.74	0.89	0.20	0.75
		B20y	0.82	0.83	0.35	0.88	0.10	0.96

Horiz	Imp	Resp	–	fund	VIX	bmf sl	bmf 3y	fed sl
9m	FED	B3m	0.01	0.01	0.04	0.01	0.04	0.06
		B3y	0.00	0.00	0.02	0.00	0.02	0.05
		B20y	0.00	0.00	0.01	0.00	0.02	0.05
	Macr	B3m			0.37	0.07	0.11	0.14
		B3y			0.50	0.08	0.02	0.16
		B20y			0.70	0.09	0.03	0.17
	θ_1	B3m	0.26	0.26	0.08	0.14	0.32	0.23
		B3y	0.20	0.20	0.02	0.11	0.89	0.17
		B20y	0.12	0.13	0.05	0.11	0.91	0.13
	θ_2	B3m	0.74	0.74	0.51	0.77	0.53	0.57
		B3y	0.80	0.80	0.47	0.80	0.07	0.63
		B20y	0.88	0.87	0.24	0.80	0.04	0.65

Horiz	Imp	Resp	–	fund	VIX	bmf sl	bmf 3y	fed sl
18m	FED	B3m	0.01	0.01	0.08	0.01	0.08	0.07
		B3y	0.00	0.00	0.03	0.00	0.03	0.05
		B20y	0.00	0.00	0.02	0.00	0.02	0.06
	Macr	B3m			0.35	0.08	0.11	0.18
		B3y			0.49	0.09	0.01	0.19
		B20y			0.69	0.10	0.02	0.20
	θ_1	B3m	0.24	0.25	0.08	0.13	0.34	0.19
		B3y	0.20	0.20	0.02	0.11	0.92	0.16
		B20y	0.19	0.20	0.05	0.11	0.93	0.14
	θ_2	B3m	0.75	0.75	0.49	0.77	0.47	0.57
		B3y	0.80	0.80	0.46	0.79	0.04	0.60
		B20y	0.81	0.80	0.24	0.79	0.03	0.61

We see in Table 4 that, clearly, the VIX has the biggest effect among the chosen macro variables, reaching up to 70% of the variation of the 20 years yield in the 18 months horizon. The FED slope and BM&F slope also explain some variation, up yo 10% and 20% respectively. On the other hand, the FED latent variable, which is highly correlated to the FED short rate, do not seem to contribute to the variance of the Brazilian sovereign yields.

The next table repeats the exercise, this time evaluating the default probabilities.

Table 5.

Variance Decomposition of the Default Probabilities. Fed 1D + Macro {0,1}-D unilateral + BR 2D models. Horizons of 1m, 9m and 18m ahead.

Hor	Imp	Resp	–	fund	VIX	bmf s	bmf 3y	fed sl
1m	FED	B3m	0.00	0.00	0.00	0.00	0.00	0.01
		B3y	0.00	0.00	0.19	0.00	0.12	0.04
		B20y	0.00	0.00	0.69	0.05	0.51	0.03
	Macr	B3m			0.54	0.05	0.08	0.04
		B3y			0.48	0.09	0.26	0.20
		B20y			0.19	0.09	0.09	0.22
	θ_1	B3m	0.22	0.22	0.09	0.15	0.21	0.34
		B3y	0.27	0.28	0.06	0.12	0.18	0.17
		B20y	0.27	0.28	0.02	0.11	0.24	0.15
	θ_2	B3m	0.78	0.77	0.37	0.80	0.70	0.62
		B3y	0.73	0.72	0.27	0.79	0.45	0.60
		B20y	0.73	0.72	0.10	0.74	0.16	0.59

Hor	Imp	Resp	–	fund	VIX	bmf s	bmf 3y	fed sl
9m	FED	B3m	0.00	0.00	0.01	0.00	0.01	0.04
		B3y	0.01	0.01	0.60	0.00	0.25	0.04
		B20y	0.01	0.01	0.92	0.06	0.65	0.03
	Macr	B3m			0.61	0.09	0.25	0.18
		B3y			0.26	0.10	0.28	0.22
		B20y			0.05	0.09	0.07	0.23
	θ_1	B3m	0.28	0.28	0.07	0.12	0.18	0.19
		B3y	0.29	0.29	0.03	0.11	0.15	0.15
		B20y	0.29	0.29	0.01	0.11	0.21	0.15
	θ_2	B3m	0.72	0.72	0.30	0.79	0.56	0.60
		B3y	0.71	0.71	0.11	0.78	0.31	0.59
		B20y	0.71	0.71	0.02	0.74	0.08	0.59

Hor	Imp	Resp	–	fund	VIX	bmf s	bmf 3y	fed sl
18m	FED	B3m	0.00	0.00	0.03	0.00	0.04	0.04
		B3y	0.01	0.01	0.73	0.01	0.37	0.04
		B20y	0.01	0.01	0.96	0.07	0.71	0.03
	Macr	B3m			0.60	0.10	0.27	0.20
		B3y			0.17	0.10	0.24	0.22
		B20y			0.03	0.09	0.05	0.23
	θ_1	B3m	0.27	0.27	0.07	0.12	0.17	0.17
		B3y	0.28	0.29	0.02	0.11	0.14	0.15
		B20y	0.28	0.29	0.00	0.11	0.20	0.15
	θ_2	B3m	0.73	0.73	0.30	0.79	0.53	0.59
		B3y	0.71	0.71	0.07	0.78	0.25	0.59
		B20y	0.71	0.71	0.01	0.73	0.05	0.59

Table 5 shows the variance decomposition of the log of the survival probabilities. Again, the VIX has the greatest effect, specially on the shorter yields. According to the model, in the 18 months horizon, it accounts for 60% of the 3-month yield, while in the models with the BM&F 3-years yield or the FED slope, it explains 27% and 20%, respectively. However, the macro-to-yield hypothesis is more restrictive for BM&F yields case, since the external yield curve cannot affect the domestic yield curve. In the next subsection, we treat domestic factors in a bilateral way.

Table 6.
Delta parameters.

	—	fund	vix	bmf sl	bmf 3y
FED	0.0037	0.0045	0.0037	0.0042	0.0065
Macro			−0.1251	0.0316	−0.0244
θ_1	0.0425	0.0433	0.0460	0.0724	−0.0661
θ_2	0.1080	0.1079	0.1197	0.0766	0.1212

Table 6 presents values of the estimated delta parameters governing the Brazilian short rate, that is $R_t = \delta_0^r + \delta_0^s + (\delta_1^r + \delta_1^s) \cdot X_t$.

5.3 Bilateral models

This subsection present the specifications with one FED latent factor, an internal and an external macro factor, and two Brazilian latent factors. The domestic macro factor has a bilateral interaction with the sovereign Brazilian factors, that is, the macro factors and the sovereign yield curves fully interact.

Table 7.
Summary of FED 1D + Macro 2D bilateral + BR 2D.

	vix bmf 3m	vix bmf sl	vix bmf 3y	vix lbov	vix lbov/ex
LL/T	52.90	52.52	52.91	55.81	55.57
#Par	51	51	51	51	51
M(In)	1.00	0.99	1.02	1.04	1.05
TU3m	2.77	2.80	3.20	3.82	3.42
TU1y	2.29	2.35	2.56	4.45	4.04
TU5y	1.02	1.04	1.03	3.20	2.65
TU10y	2.29	2.40	2.12	1.77	1.45
c(θ_1,s)	−0.08	−0.48	−0.04	−0.86	−0.86
c(θ_1,l)	−0.45	−0.02	−0.52	0.50	0.50
c(θ_2,s)	−0.54	−0.54	−0.55	−0.63	−0.64
c(θ_2,l)	0.92	0.93	0.93	0.96	0.96
M(σ_u)	61	62	61	67	68

Table 7 is a summary of the main models. Again, the likelihood indicates that more information was added with the inclusion of more macro factors. The out-of-sample forecasting performance continues to be low. The in sample fitting improved a little. The unobservable factor θ_2 can again be interpreted as the level.

Table 8.
Variance Decomposition. Fed 1D + Macro 2D + BR2D models.

Hor	Imp	Resp	vx bmf3m	vx bmfsl	vx bmf3y	vx lbov	vx lbov/ex
1m	FED	B3m	0.01	0.00	0.01	0.01	0.01
		B3y	0.00	0.00	0.00	0.00	0.00
		B20y	0.00	0.00	0.00	0.00	0.00
	VIX	B3m	0.04	0.02	0.04	0.02	0.02
		B3y	0.05	0.02	0.05	0.04	0.04
		B20y	0.20	0.27	0.20	0.34	0.36
	bmfbov	B3m	0.00	0.02	0.01	0.00	0.00
		B3y	0.00	0.03	0.00	0.00	0.00
		B20y	0.00	0.03	0.01	0.00	0.00
	θ_1	B3m	0.19	0.38	0.17	0.68	0.68
		B3y	0.03	0.13	0.02	0.20	0.19
		B20y	0.00	0.02	0.00	0.05	0.05
	θ_2	B3m	0.19	0.38	0.17	0.68	0.68
		B3y	0.91	0.81	0.93	0.76	0.76
		B20y	0.80	0.67	0.79	0.61	0.59

Hor	Imp	Resp	vx bmf3m	vx bmfsl	vx bmf3y	vx lbov	vx lbov/ex
9m	FED	B3m	0.02	0.02	0.03	0.02	0.03
		B3y	0.01	0.01	0.01	0.04	0.05
		B20y	0.00	0.00	0.00	0.04	0.04
	VIX	B3m	0.26	0.20	0.32	0.21	0.21
		B3y	0.33	0.23	0.39	0.20	0.21
		B20y	0.48	0.38	0.53	0.15	0.16
	bmfbov	B3m	0.02	0.07	0.01	0.00	0.00
		B3y	0.02	0.10	0.01	0.03	0.01
		B20y	0.01	0.09	0.02	0.02	0.00
	θ_1	B3m	0.12	0.22	0.09	0.51	0.50
		B3y	0.04	0.09	0.01	0.50	0.52
		B20y	0.01	0.04	0.01	0.69	0.70
	θ_2	B3m	0.57	0.48	0.56	0.25	0.25
		B3y	0.61	0.57	0.58	0.23	0.21
		B20y	0.50	0.49	0.43	0.10	0.09

Hor	Imp	Resp	vx bmf3m	vx bmfsl	vx bmf3y	vx lbov	vx lbov/ex
18m	FED	B3m	0.06	0.06	0.06	0.08	0.11
		B3y	0.03	0.03	0.03	0.07	0.08
		B20y	0.01	0.01	0.01	0.05	0.06
	VIX	B3m	0.26	0.20	0.31	0.20	0.20
		B3y	0.32	0.23	0.39	0.10	0.10
		B20y	0.47	0.38	0.53	0.07	0.07
	bmfbov	B3m	0.02	0.07	0.01	0.01	0.00
		B3y	0.02	0.10	0.01	0.03	0.01
		B20y	0.01	0.09	0.02	0.02	0.00
	θ_1	B3m	0.12	0.21	0.08	0.48	0.46
		B3y	0.04	0.09	0.01	0.71	0.73
		B20y	0.01	0.04	0.01	0.81	0.83
	θ_2	B3m	0.55	0.46	0.53	0.23	0.23
		B3y	0.59	0.56	0.56	0.10	0.09
		B20y	0.50	0.48	0.43	0.04	0.04

Table 8 show the variance decomposition of {1m,3y,20y}-yields for forecast horizons of {1,9,18}-months ahead of our main models. In line with the preliminary models, the VIX is again the most macro important factor influencing the yields. Of the domestic yields, only the 3y-1m slope has some effect.

Next, in Table 9, we present the variance decomposition of the default probabilities of the main models. In the 9 and 18 month horizon decomposition (free from the initial condition effects) the results show that:

- In all specifications, the FED has small effect of short bonds, but about 90% of changes in implied default probabilities of longer bonds are attributable to changes in the FED short rates.
- In all specifications, the VIX has small effect of long bonds, but about 50% of changes in implied default probabilities of shorter bonds are attributable to changes in the VIX index.
- Of the domestic factors, only the slope of the term structure has some effect, account for 11% of changes in implied probabilities of the short bond.
- The domestic short and long rate and the stock exchange index could not be traced as a source of default probability changes.

Figure 3 shows the evolution of the 1-yearl survival probability along the sample, and Figure 4 the term structure of default probabilities in the last day of the sample. The figure indicates some robustness of the estimations.

Impulse response functions are plotted after the default probabilities. Each figure presents the effect of a shock of one standard deviation of a monthly variation of one state variable. Figure 5 evaluate the impact of a FED shock on itself, on the macro factors and on the {3m, 3y, 20y}-yields. The next figure shows the response to one deviation of a monthly variation of VIX shocks. All the yield rates are increased about 1% in absolute terms 3 months after the shock an then decreases. Figure 7 shows the impact of the domestic macro factors. Changes in either the domestic short or long rate did not result in changes of the sovereign yields. But the domestic slope did cause an increase. It may indicate a change of expectations due to a future rise in inflation. A rise of the domestic stock exchange caused a small decrease of the yields.

Figure 10 shows the impact of an increase of one deviation of a monthly variation of the FED latent factor (approximately the FED short rate) on the default probabilities. It shows that the survival probability fall by up to 2% in relative terms. An increase in VIX also decreases the survival probability, but about 0.6% in relative terms. Of the domestic factors, only the BM&F slope has some impact, decreasing the long end survival probability by about 0.35% in relative terms. Factor Loadings of the model are shown in Figure 15.

Table 9.
Variance Decomposition of the Default Probabilities. Fed 1D + Macro 2D + BR 2D models.

Hor	Imp	Resp	vx bmf3m	vx bmfsl	vx bmf3y	vx lbov	vx lbov/ex
1m	FED	B3m	0.00	0.00	0.00	0.02	0.02
		B3y	0.08	0.07	0.09	0.10	0.21
		B20y	0.51	0.47	0.52	0.66	0.80
	VIX	B3m	0.29	0.19	0.34	0.15	0.16
		B3y	0.36	0.31	0.55	0.29	0.26
		B20y	0.20	0.18	0.29	0.11	0.07
	bmfbov	B3m	0.01	0.08	0.01	0.00	0.00
		B3y	0.03	0.11	0.02	0.01	0.01
		B20y	0.01	0.06	0.01	0.00	0.00
	θ_1	B3m	0.11	0.21	0.06	0.61	0.61
		B3y	0.09	0.11	0.01	0.45	0.38
		B20y	0.05	0.06	0.00	0.18	0.10
	θ_2	B3m	0.11	0.21	0.06	0.61	0.61
		B3y	0.43	0.41	0.33	0.16	0.14
		B20y	0.23	0.24	0.18	0.06	0.03

Hor	Imp	Resp	vx bmf3m	vx bmfsl	vx bmf3y	vx lbov	vx lbov/ex
9m	FED	B3m	0.00	0.00	0.00	0.02	0.02
		B3y	0.29	0.21	0.29	0.42	0.59
		B20y	0.79	0.73	0.78	0.88	0.93
	VIX	B3m	0.42	0.32	0.56	0.31	0.31
		B3y	0.33	0.31	0.51	0.25	0.18
		B20y	0.10	0.11	0.16	0.05	0.03
	bmfbov	B3m	0.03	0.11	0.01	0.00	0.00
		B3y	0.03	0.10	0.02	0.01	0.01
		B20y	0.01	0.03	0.01	0.00	0.00
	θ_1	B3m	0.09	0.12	0.03	0.49	0.49
		B3y	0.07	0.06	0.00	0.24	0.16
		B20y	0.02	0.02	0.00	0.05	0.03
	θ_2	B3m	0.46	0.44	0.39	0.18	0.18
		B3y	0.29	0.31	0.17	0.08	0.06
		B20y	0.08	0.11	0.05	0.02	0.01

Hor	Imp	Resp	vx bmf3m	vx bmfsl	vx bmf3y	vx lbov	vx lbov/ex
18m	FED	B3m	0.01	0.01	0.02	0.04	0.06
		B3y	0.45	0.35	0.44	0.61	0.75
		B20y	0.88	0.83	0.87	0.93	0.96
	VIX	B3m	0.42	0.32	0.56	0.30	0.30
		B3y	0.25	0.26	0.40	0.17	0.11
		B20y	0.06	0.07	0.09	0.03	0.02
	bmfbov	B3m	0.03	0.11	0.01	0.00	0.01
		B3y	0.03	0.08	0.02	0.00	0.01
		B20y	0.01	0.02	0.00	0.00	0.00
	θ_1	B3m	0.09	0.12	0.03	0.48	0.47
		B3y	0.05	0.05	0.00	0.16	0.10
		B20y	0.01	0.01	0.00	0.03	0.01
	θ_2	B3m	0.45	0.43	0.38	0.17	0.17
		B3y	0.22	0.26	0.14	0.06	0.03
		B20y	0.05	0.07	0.03	0.01	0.00

Table 10. The estimated δ_1 parameters of the bilateral models.

	vx bmf sl	vx bmf 3y	vx lbov	vx lbov/ex
FED	0.0076	0.0087	−0.0130	−0.0126
VIX	−0.0242	−0.0327	−0.0772	−0.0739
BMF/Bov	0.0035	0.1164	0.0981	0.1005
θ_1	0.0769	0.0617	0.1110	0.1110
θ_2	0.0877	0.1073	0.0665	0.0665

5.4 One step estimation

In this subsection, we show how to make a simultaneous estimation of the US and the Brazilian yield curve and the macro factors. The difference is that we estimate the parameters in one step instead of the usual two-step (or even 3-step) procedure. Since we suppose that the US yield curve parameters are not affected by the Brazilian parameters, the joint probability density of the yield curves and macro factors can be decomposed:

$$f(Y^{US}, Y^{BR}, M^{US}, M^{BR}; \Psi^{US}, \Psi^{BR}) = f(Y^{US}, M^{US}; \Psi^{US}) f(Y^{BR}, M^{BR}; \Psi^{BR} | Y^{US}, M^{US}; \Psi^{US}). \tag{77}$$

Thus, the loglikelihood will be the sum of two functions, one depending on Ψ^{US} and the other on (Ψ^{US}, Ψ^{BR}).

If we stick to models with one FED latent factor, the VIX as the sole macro factor, and one Brazilian latent factor, there are still many possible specifications depending on choices of the parameters δ, K, and Σ of the FED and Brazilian curves.

In our previous models, the US short rate did not depended on macro factors, that is $r_t = \delta_0^r + (\delta_1^r, 0, 0) \cdot X_t$. It clearly does not depend on the Brazilian macro factor, but what can be said with respect to the VIX? Using $r_t = \delta_0^r + (\delta_1^r, \delta_2^r, 0) \cdot X_t$ can answer the question.

Likewise, the mean reversion matrix K of the FED curve did not depend on the macro factors. The dynamics of the specification will change completely depending on the choice of

$$K = \begin{pmatrix} K_{\theta\theta} & K_{\theta M} \\ K_{M\theta} & K_{MM} \end{pmatrix} \text{ or } K = \begin{pmatrix} K_{\theta\theta} & 0 \\ K_{M\theta} & K_{MM} \end{pmatrix}. \tag{78}$$

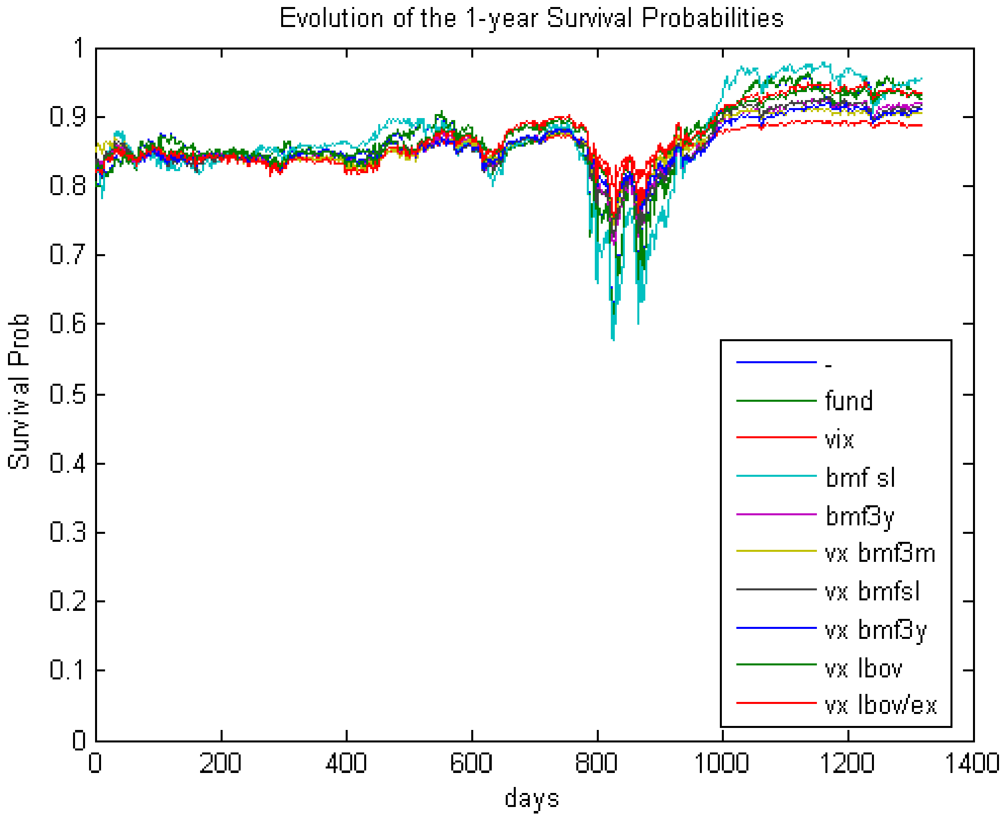

Figure 3:

What will happen if we insert a $K_{\theta M}$ factor in the FED curve? The model will become a bilateral model with respect to VIX. Also, the FED volatility matrix Σ can be the unity matrix or

$$\Sigma = \begin{pmatrix} \mathbb{I} & 0 \\ \Sigma_{M\theta} & \Sigma_{MM} \end{pmatrix}. \tag{79}$$

The choice is between having latent shocks contemporaneous to macro shocks or not. Similarly, modeling of the Brazilian curve involves choices of parameters which will affect the dynamics of the state vectors. However, the options are limited by the identification restrictions, and we leave those questions for future work.

5.5 Default Probabilities

Figure 3 depicts the path of 1 minus the probability of a default occurring before 1 year, that is, the market implied probability that the bond will survive for another year. It can be seen that different specifications tend to present similar probabilities. Figure 4 depicts the term structure of default survival probabilities for the last day of the sample.

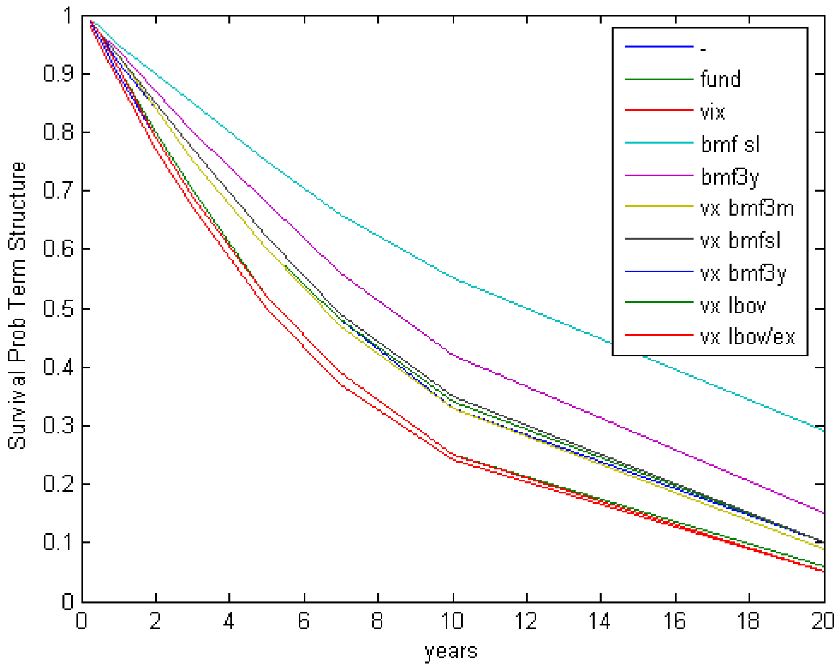

Figure 4: Term Structure of Survival Probabilities in the last day of the sample.

6 Conclusion

This article proposed an approach combining term structure models with macro factors and reduced credit risk models, aiming to measure how unexpected macroeconomic changes affect sovereign default probabilities. Amato and Luisi (2005) also explore the same ideas with respect to corporate credit risk, but our article uses a fully interacting dynamics, in which macro factors afect and are affected by the credit spreads. Also, we presented and estimated an identified model, while other articles use super or sub-identified models.

We tested the influence of two domestic macro factors and term structure on the sovereign term structure of interest rates and of credit spreads, and the result was that VIX and FED had greater impact. We calculated variance decompositions and impulse response functions in order to make quantitative predictions. The model presented good fitting to data, but did not show good forecasting performance. Our results have shown that VIX is an important factor for the default probabilities of emerging market short-term bonds. On the other hand, the FED is an important indicator for the longer yields.

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A Appendix: Identification

In order to identify the unobservable factors θ, we modify the method introduced by Dai and Singleton (2000) to the case with macro factors. There are several possibilities and only one choice is implemented. The identification is necessary because not all parameters can be estimated. There are linear transformations on the parameter space leaving the short rate, and thus the yields, constant. These transformations can be considered degrees of freedom that must be spent so that the model becomes identified. If a model is not identified, as Collin-Dufresne et al (2006) put it, two researchers using the same data can arrive at different sets of parameter estimates even if they succeed to maximize. Also, the impulse response functions could be arbitrarily changed and model forecasts would be meaningless. Using the notation of the continuous time specification, in which $\Psi = (\delta_0, \delta_1, K, \xi, \lambda_0, \lambda_1, \Sigma)$, the affine invariant transformation T is defined by a nonsingular matrix L such that the

$$T_L(\Psi) = (\delta_0, (L^{\intercal})^{-1}\delta_1, LKL^{-1}, L\xi, \lambda_0, \lambda_1, L\Sigma). \tag{80}$$

Another invariant transformation is the Brownian motion rotation O, which takes a vector of unobserved, independent Brownian motions into another vector of independent Brownian motions:

$$T_O(\Psi) = (\delta_0, \delta_1, K, \xi, O\lambda_0, O\lambda_1, \Sigma O^{\intercal}). \tag{81}$$

The rotations do not affect the state factors and can always be used to make Σ a triangular matrix. We impose a lower triangular Σ, which implies that macro factors do not react contemporaneously to monetary policy.

We impose $E(\theta) = 0$ as Dai and Singleton (2000), and subtract the mean value of the macro factors, so that $E(M) = 0$.Then, $\xi = 0$. Also, in contrast to the case with purely latent factors, the transformations L must preserve the macro factors:

$$L = \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}, \tag{82}$$

The matrices A and B matrix are chosen such that $Var(\theta) = \mathbb{I}$, $\sum_{M\theta} = 0$, that is

$$L\Sigma = \begin{pmatrix} \Sigma_{MM} & 0 \\ 0 & I \end{pmatrix} \tag{83}$$

where, as said before, Σ_{MM} is lower triangular. This implies that macro and monetary factors do not have correlated contemporaneous innovations.

There is another invariant transformation that must be used in case $L\Sigma$ has the special format shown above. There is another operator

$$R = \begin{pmatrix} I & 0 \\ 0 & O \end{pmatrix}, \tag{84}$$

where O is another rotation which makes $\Phi_{\theta\theta}$ lower triangular. Summing up, we have

$$\Sigma = \begin{pmatrix} \Sigma_{MM} & 0 \\ 0 & I \end{pmatrix}, \quad \xi = 0, \quad \Phi = \begin{pmatrix} \Phi_{MM} & \Phi_{M\theta} \\ \Phi_{\theta M} & \Phi_{\theta\theta} \end{pmatrix} \text{ with } \Phi_{\theta\theta} \text{ lower triangular.} \tag{85}$$

This completes an exactly identified specification.

We remark that another possibility is imposing a triangular $\Phi^{\star}_{\theta\theta}$ instead of $\Phi_{\theta\theta}$. Actually, many other identifications are possible. Duffie and Kan (1996) and Collin-Dufresne et al (2006) propose an identification strategy that is different from Dai and Singleton (2000) and ours.

B Figures

C Estimated Parameters and Confidence Intervals.

Table 10.

Estimated parameters and 1-sd confidence intervals of the main models. Part I.

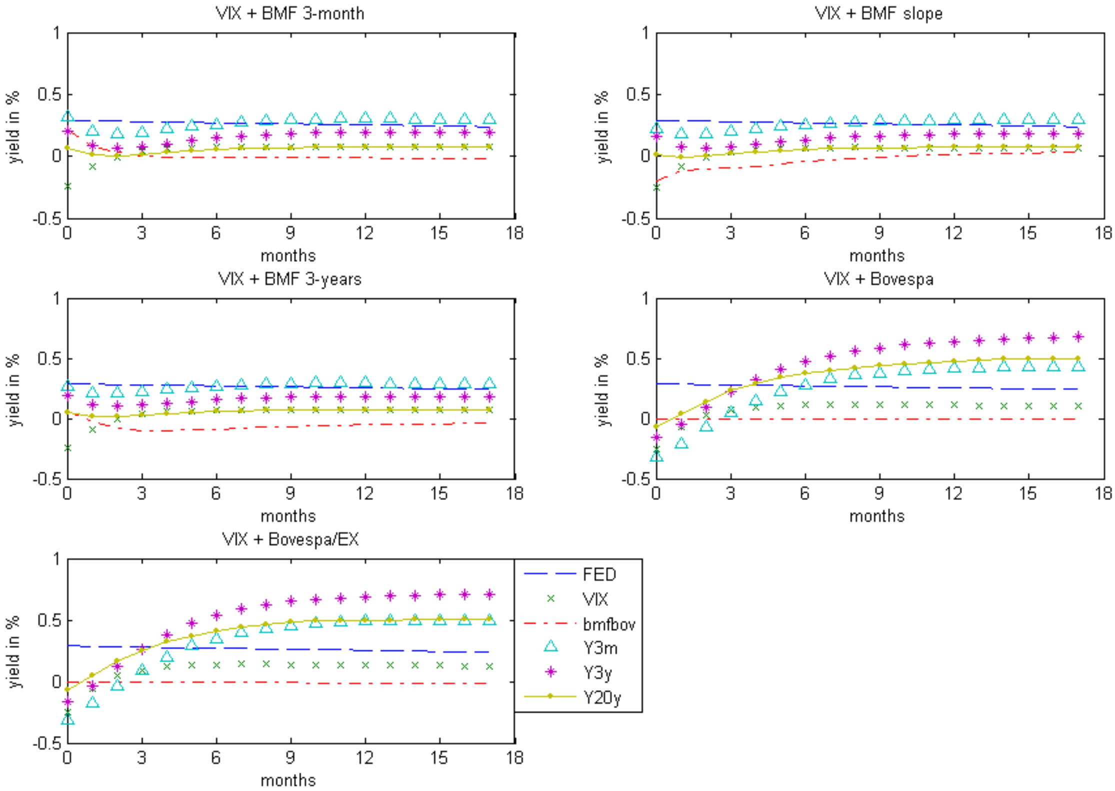

Figure 5: Impulse Response: FED shock.

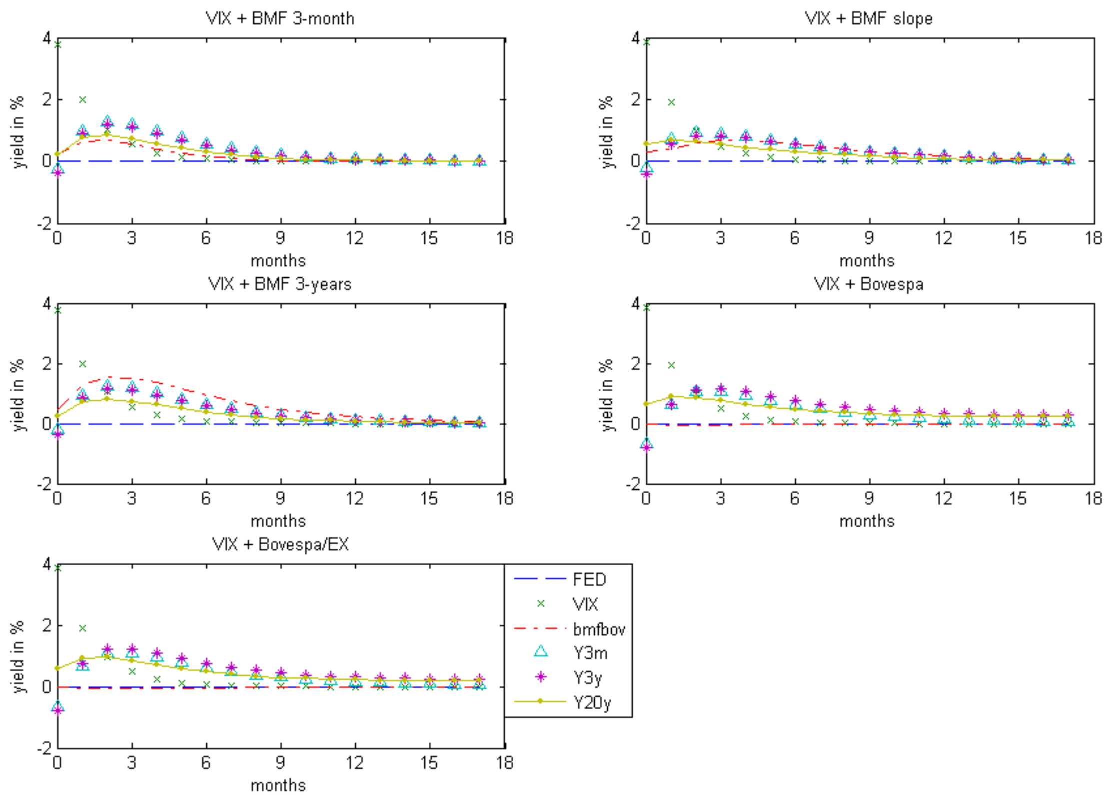

Figure 6: Impuse Response: VIX shock.

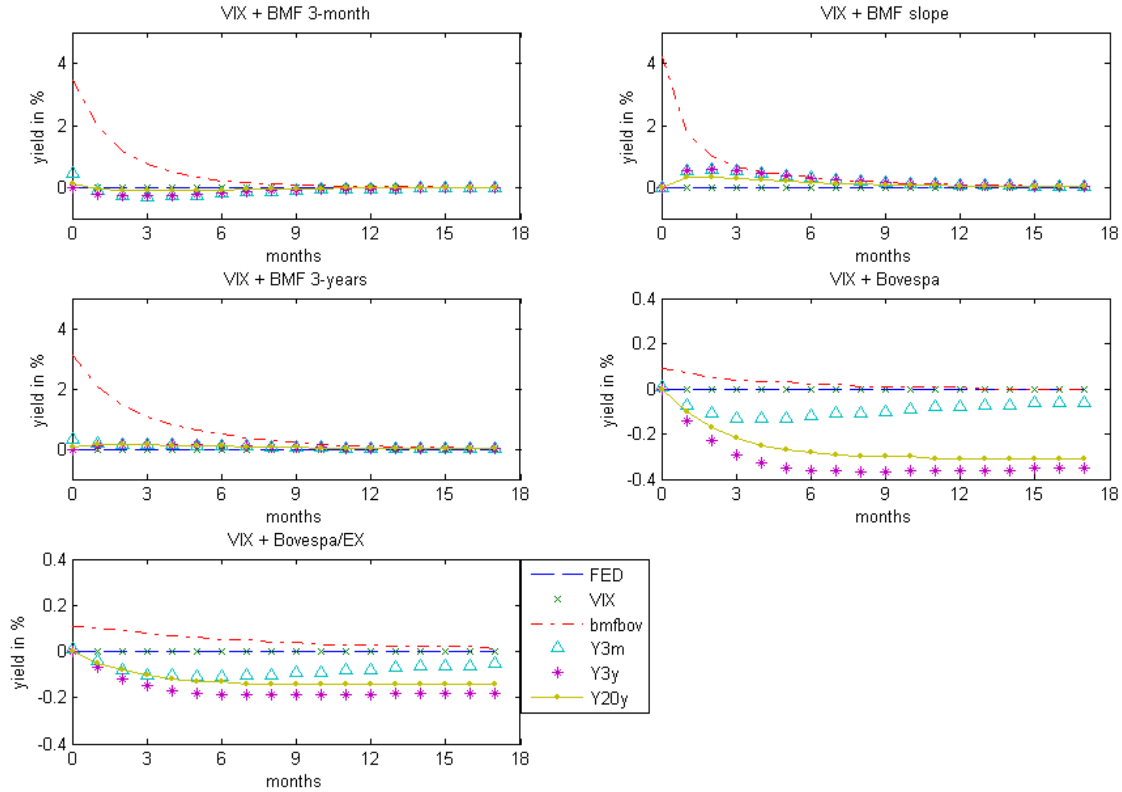

Figure 7: Impulse Response: BM&F or Bovespa shock.

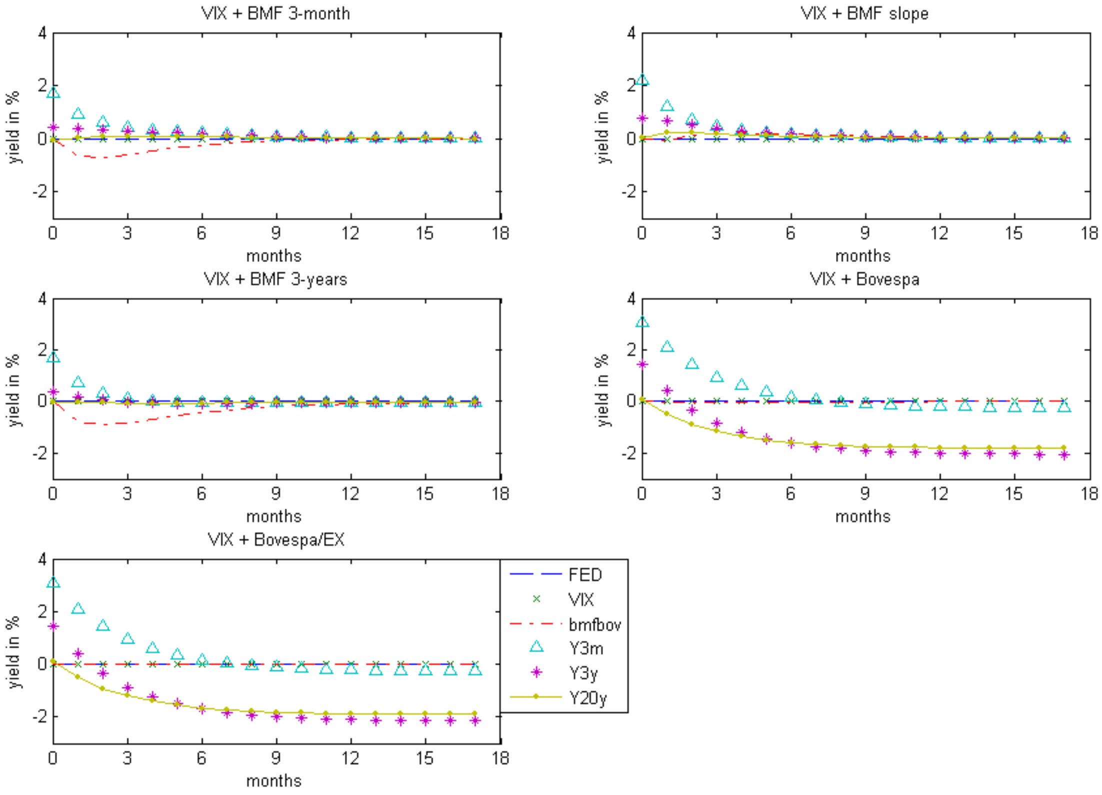

Figure 8: Impulse Response: unobservable factor θ_1 shock.

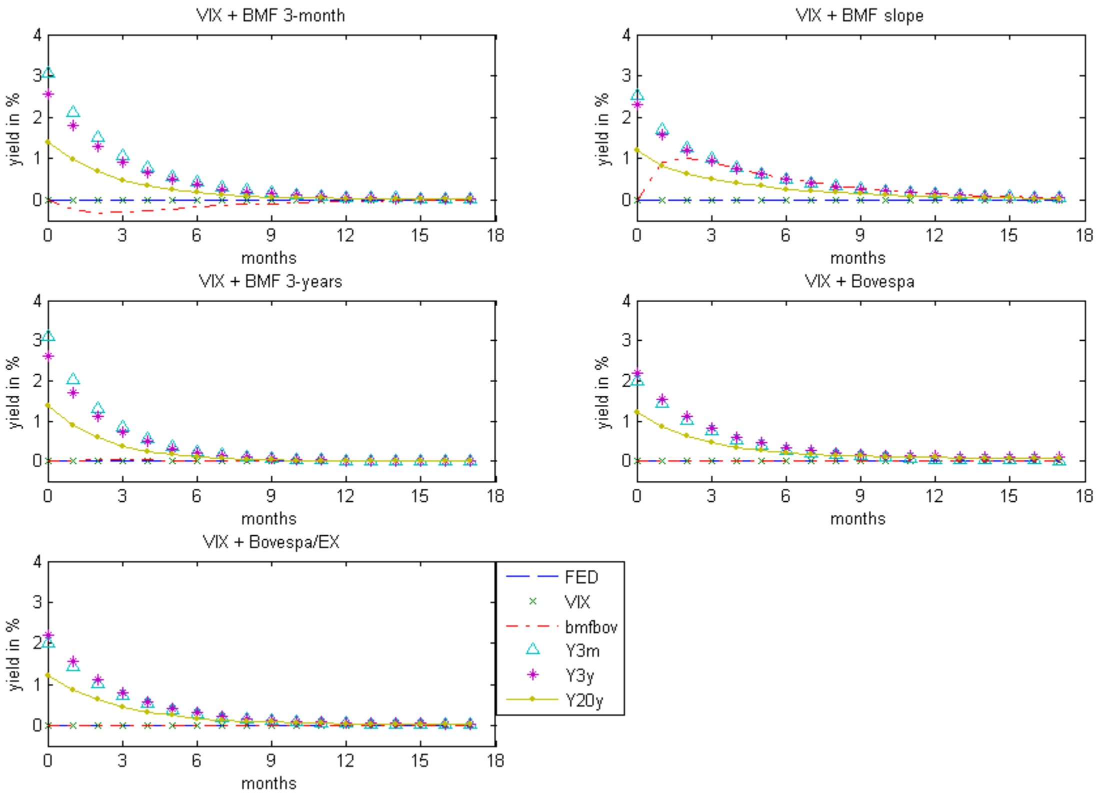

Figure 9: Impulse Response: unobservable factor θ_2 (or level) shock.

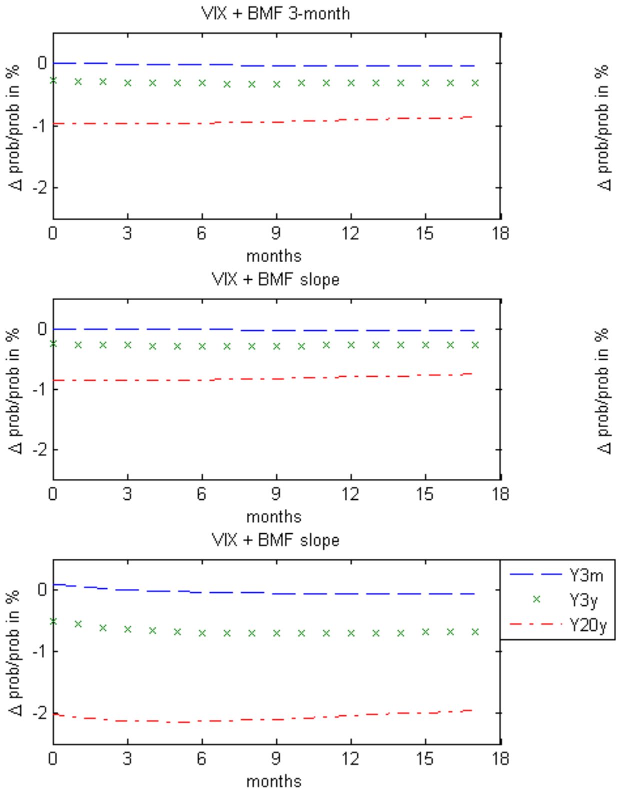

Figure 10: Impulse Response Survival Probabilties: FED shock.

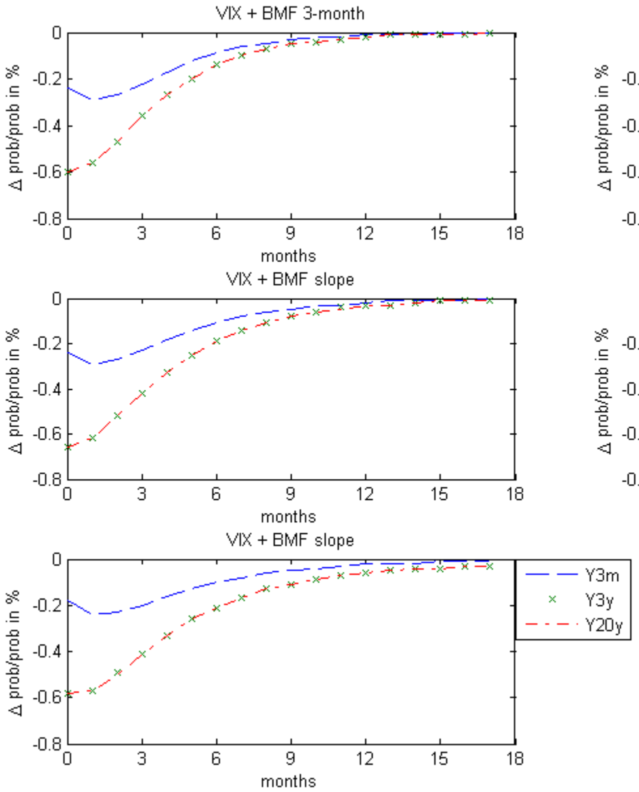

Figure 11: Impulse Response Survival Probabilities: VIX shock.

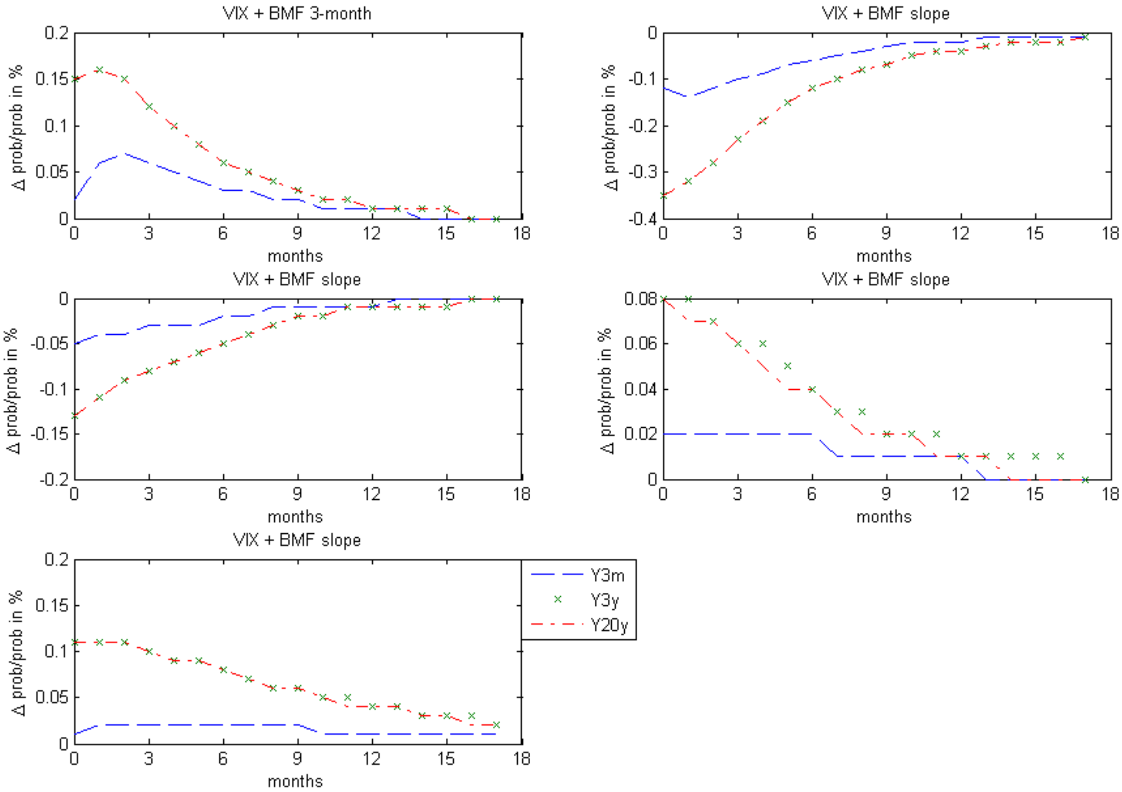

Figure 12: Impulse Response Survival Probabilities: BM&F or Bovespa shocks.

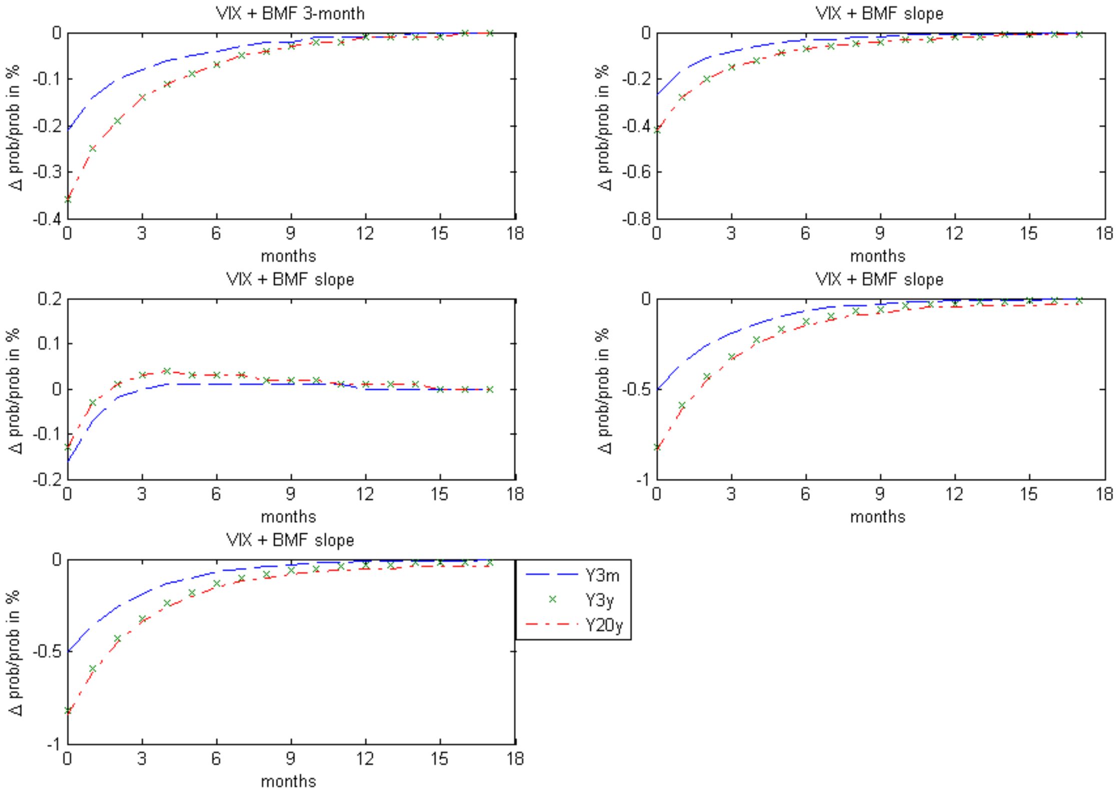

Figure 13: Impulse Response Survival Probabilities: θ_1 shock.

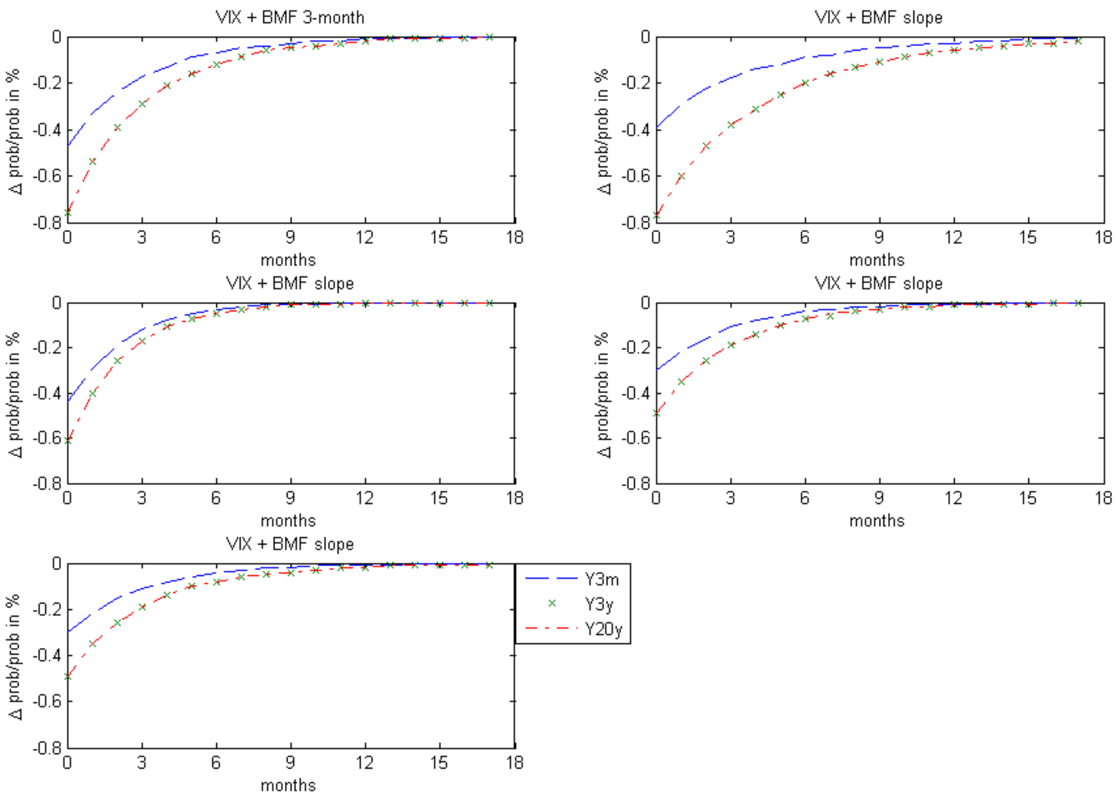

Figure 14: Impulse Response Survival Probabilities: θ_2 (or level) shock.

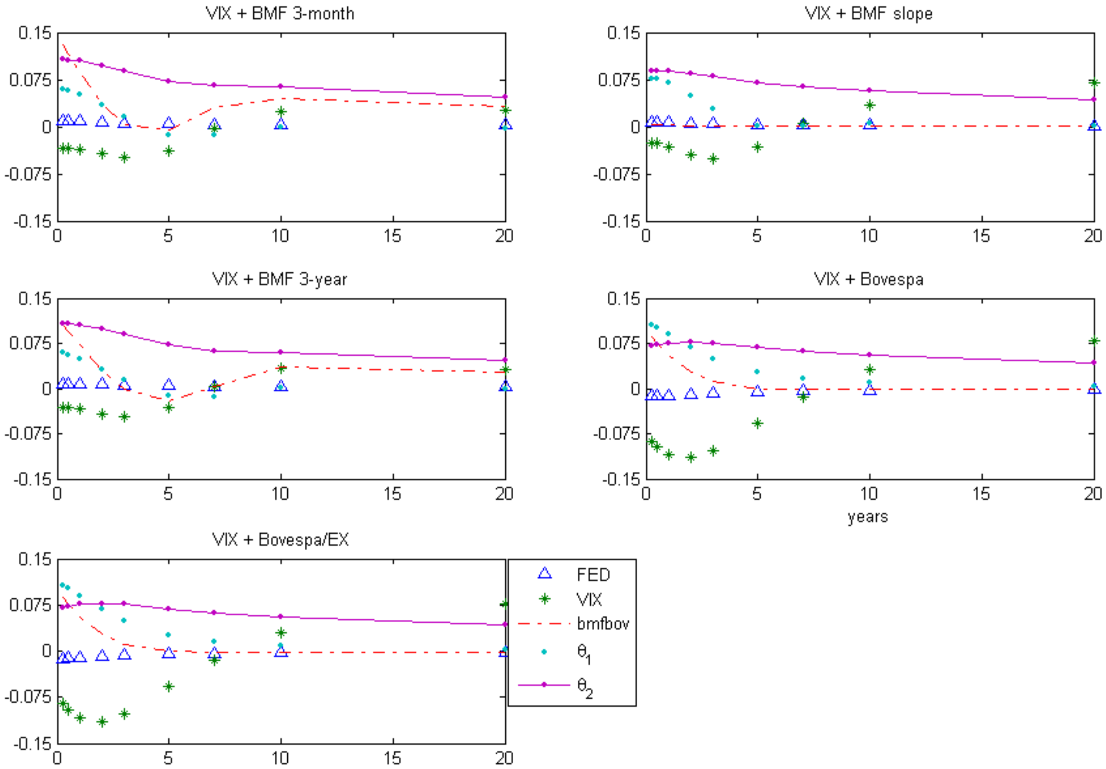

Figure 15: Factor Loadings of the main models.

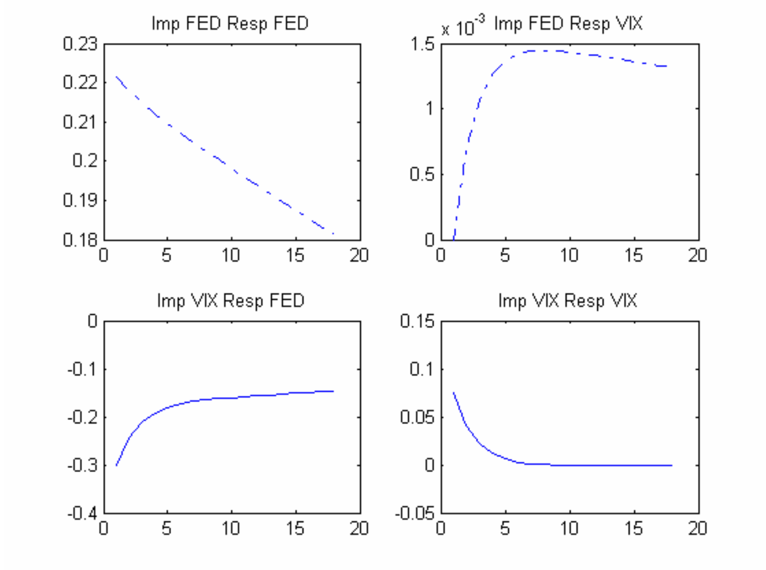

Figure 16: Impulse response 1. Sub identified model with FED and VIX. Parameter estimation 1.

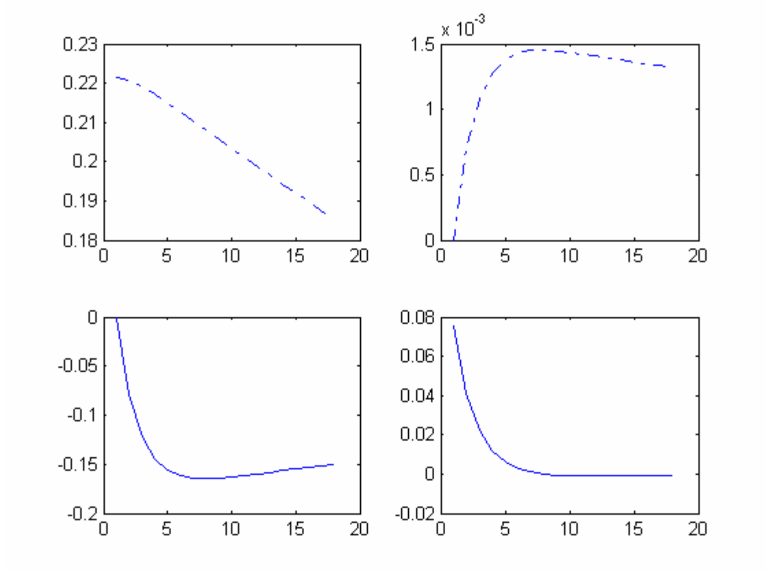

Figure 17: Impulse response 2. Sub identified model with FED and VIX. Parameter estimation 2.

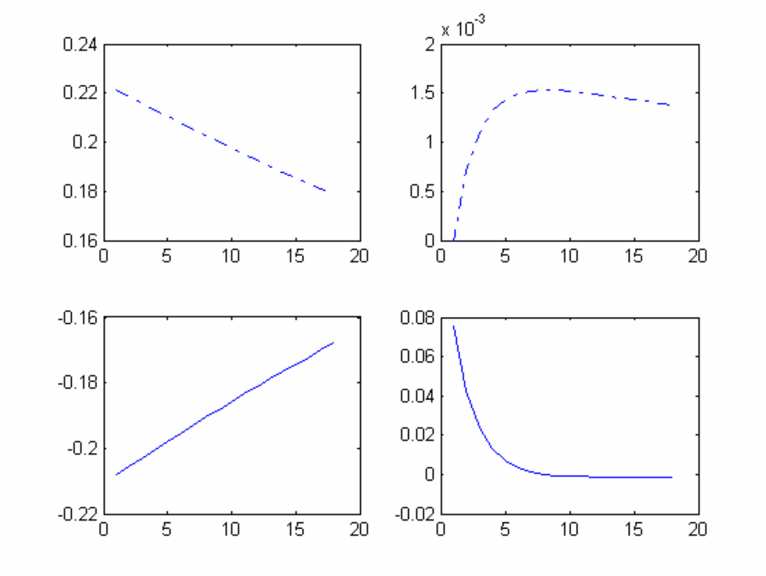

Figure 18: Impulse response 3. Sub identified model with FED and VIX. Parameter estimation 3.

	vix+bmf3m		vix+bmfsl		vix+bmf3y		vix+bovespa		vix+bovespa/ex	
	Par	sd	Par	sd	Par	sd	Par	sd	Par	sd
δ_1	0.0015	0.0024	0.0076	0.0017	0.0087	0.0024	-0.0130	0.0016	-0.0123	0.0014
δ_2	-0.0617	x	-0.0242	x	-0.0327	0.0085	-0.0772	0.0063	-0.0742	0.0052
δ_3	0.0005	0.0022	0.0035	0.0001	0.1164	0.0032	0.0981	x	0.1007	0.0000
δ_4	-0.0255	x	0.0769	0.0017	0.0617	0.0017	0.1110	0.0000	0.1109	0.0000
δ_5	0.1233	0.0038	0.0877	0.0013	0.1073	0.0021	0.0665	0.0000	0.0665	0.0000
a_2^*	-3.3912	0.0188	0.1176	0.0060	0.4656	0.0424	0.2375	0.0059	0.2402	0.0054
a_3^*	2.7542	0.0155	-15.13	0.6152	-0.5998	0.0457	-0.751	0.2494	-0.1889	0.7477
a_4^*	-0.3910	0.0392	0.4050	0.0327	0.6535	0.0718	-171.4	1.4626	-184.8	0.0344
a_5^*	-1.5964	0.0300	0.4790	0.0237	0.5361	0.0298	0.4344	0.0257	0.4341	0.0249
K^{21}	0.0734	0.0016	-0.0039	0.0006	-0.0069	0.0024	0.0045	0.0010	0.0044	0.0010
K^{22}	0.2789	0.0249	0.1170	0.0174	0.6570	0.0434	0.2469	x	0.2517	0.0078
K^{32}	0.2364	0.0418	-26.94	2.7211	-1.0128	0.0849	0.0054	x	0.0414	0.0291
K^{33}	0.0358	0.0010	-0.6638	x	-0.5834	0.0022	0.3593	0.0065	0.3537	0.0168
K^{34}	-0.3428	0.0076	-29.40	0.6891	-0.4074	0.0025	-0.5228	0.0002	-0.5232	0.0018
K^{35}	-0.6996	0.0054	-3.686	0.1492	-0.1539	0.0059	-0.3102	0.0006	-0.3099	0.0007
K^{41}	0.0125	0.0023	0.0021	0.0025	-0.0020	0.0041	0.0048	0.0071	0.0045	0.0072
K^{42}	0.4319	0.0248	1.4733	0.0914	1.8553	0.0429	1.4873	0.0000	1.4974	0.0422
K^{43}	0.4906	x	0.0450	0.0008	1.7020	0.0383	0.5742	0.0000	0.5996	0.0009
K^{44}	0.1264	0.0102	0.9163	0.0154	0.5615	0.0085	0.7363	0.0005	0.7441	x
K^{51}	0.0527	0.0049	0.0045	0.0035	0.0016	0.0047	-0.0182	0.0055	-0.0167	0.0055
K^{52}	0.6747	0.0419	-0.1420	0.0301	0.1045	0.0332	-0.7765	x	-0.7520	0.0301
K^{53}	0.5595	0.0076	0.0086	x	0.4462	0.0001	0.0943	0.0000	0.0937	0.0010
K^{54}	0.0734	0.0046	0.3348	0.0021	0.2810	0.0032	0.0735	0.0000	0.0736	0.0000
K^{55}	0.0045	0.0052	0.1181	0.0008	0.1613	0.0026	0.0451	0.0000	0.0451	0.0000

Table 11.
Estimated parameters and 1 standard deviation (sd) confidence intervals of the main models. Part II.

	vix+bmf3m		vix+bmfsl		vix+bmf3y		vix+bovespa		vix+bovespa/ex	
	Par	sd	Par	sd	Par	sd	Par	sd	Par	sd
K^{*21}	−0.053	0.0438	−0.0457	0.0443	−0.0437	0.0432	−0.0697	0.0415	−0.0843	0.0434
K^{*22}	7.4784	2.1548	8.3093	1.7531	7.6871	1.9719	8.3080	1.7198	8.2697	1.8194
K^{*32}	−1.383	1.2890	0.0000	1.2961	−2.2925	1.1346	0.0507	0.0191	0.0407	0.0278
K^{*33}	6.4031	1.3427	11.904	2.1064	5.5400	1.7209	2.9970	0.7565	1.5462	0.5366
K^{*34}	0.1163	0.2454	0.2154	0.2919	0.6059	0.2460	0.0048	0.0000	0.0044	x
K^{*35}	0.1520	0.1702	−0.7484	0.1570	−0.0134	0.1781	0.0029	0.0000	0.0026	x
K^{*41}	−0.712	0.1961	−0.4406	0.1795	−0.3843	0.2089	0.0000	0.1678	0.0006	0.1681
K^{*42}	40.712	1.2936	1.6728	8.2186	15.236	7.6095	−2.2123	6.5458	0.0064	10.506
K^{*43}	−6.785	5.2695	11.927	8.5745	24.583	3.1266	−149.8	8.0909	−45.04	53.709
K^{*44}	5.7131	1.8204	12.246	1.7345	9.2538	1.7525	−0.0008	0.0019	−0.0008	0.0017
K^{*51}	−1.363	0.2158	0.1698	0.2116	0.0920	0.2370	−1.362	0.2660	−1.4139	0.1167
K^{*52}	−36.68	0.0000	−24.97	2.4255	−31.04	0.4505	−32.406	31.6276	−38.07	16.503
K^{*53}	7.312	1.4328	−43.84	34.704	−10.85	11.167	420.31	26.9421	143.58	13.806
K^{*54}	2.873	2.2072	−4.439	2.1716	−0.268	2.3170	6.8271	0.8592	6.7689	1.5567
K^{*55}	12.963	1.3311	5.6504	0.7062	5.1173	2.2031	4.0883	0.5145	4.0605	0.9341
Σ^{21}	−0.018	0.0071	−0.0173	0.0073	−0.0167	0.0070	−0.0177	0.0072	−0.0179	0.0072
Σ^{22}	0.2605	0.0056	0.2659	0.0056	0.2610	0.0055	0.2659	0.0052	0.2657	0.0050
Σ^{31}	0.0075	0.0033	−0.0068	0.0040	0.0020	0.0030	0.0000	0.0001	0.0000	0.0001
Σ^{32}	0.0073	0.0035	0.0105	0.0042	0.0163	0.0031	−0.0011	0.0001	−0.0013	0.0001
Σ^{33}	0.1189	0.0024	0.1457	0.0028	0.1080	0.0022	0.0030	0.0000	0.0038	0.0000
σ_1	0.0021	0.0000	0.0021	0.0000	0.0022	0.0001	0.0021	0.0000	0.0021	0.0000
σ_2	0.0053	0.0001	0.0053	0.0001	0.0055	0.0001	0.0058	0.0001	0.0058	0.0001
σ_3	0.0087	0.0002	0.0090	0.0002	0.0091	0.0002	0.0099	0.0002	0.0099	0.0002
σ_4	0.0082	0.0002	0.0079	0.0002	0.0079	0.0002	0.0083	0.0001	0.0083	0.0002
σ_5	0.0046	0.0001	0.0050	0.0001	0.0045	0.0001	0.0054	0.0001	0.0054	0.0001
σ_6	0.0077	0.0001	0.0074	0.0002	0.0070	0.0001	0.0081	0.0002	0.0081	0.0002
σ_7	0.0074	0.0001	0.0068	0.0001	0.0066	0.0001	0.0076	0.0001	0.0076	0.0002

Table 12. Parameters of the FED 1D:

δ_1	a^*	K^*	K
0.0078	0.9898	0.2341	0.1325

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