178
DISCUSSION PAPER

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IDENTIFICATION OF AFFINE TERM STRUCTURE MODELS WITH OBSERVED FACTORS: ECONOMIC SHOCKS ON BRAZILIAN YIELD CURVES

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178

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IDENTIFICATION OF AFFINE TERM STRUCTURE MODELS WITH OBSERVED FACTORS: ECONOMIC SHOCKS ON BRAZILIAN YIELD CURVES

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DISCUSSION PAPER

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SINOPSE

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ABSTRACT

We propose different exactly identified specifications of affine models with observed macri factors. The models are compared estimating Brazilian domestic and sovereign yield curves.

SUMMARY

Identification of Affine Term Structure Models with Observed Factors: Economic Shocks on Brazilian Yield Curves

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March 2007

Abstract

We propose different exactly identified specifications of affine models with observed macro factors. The models are compared estimating Brazilian domestic and sovereign yield curves.

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1 Introduction

The fundamental works of Vasicek and of Cox, Ingersoll and Ross on term structure models developed one state variable models. Multifactor affine term structure models generalized those models, enhancing the goodness of fit and the forecasting performance, as well as permitting the inclusion of default and of macro factors as state variables.

Certainly, the inclusion of more variables results in a higher dimension of the problem that will increase the computational burden at the stage of inference. But a not so clear issue raised by the multifactor models is the identification of the parameters. Not all can be estimated, and some can be set to any value.

Since decreasing the number of parameters is an immediate way to ease the estimation of the parameters, many authors impose restrictions not always justified by economic reasons. But arbitrary restrictions may over-identify one part while other parameters may remain unidentified. We show a well defined set of alternative restrictions which are necessary for identification of affine models with macro factors.

There are two main approaches to identification: Dai and Singleton (DS, 2000), which uses "invariant" transformations to rotate the model to a canonical

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specification, and Duffie and Kan (1996) and Collin-Dufresne et al (2006), which use a specific rotation to observed state variables.

We apply DS transformations in Ang and Piazzesi (2003, AP) model added with credit spreads. DS showed that there are transformations of the parameter space associated to linear operators that preserve the short rate and the premium process, and hence all yields. Since more than one vector of parameter is associated to the observed yields, we have the freedom to choose a specification in which some parameters are set to zero or other fixed value.

Moreover, unidentified impulse response functions vary and cannot be used to interpret the consequences of shocks in the state variables on the yield curve. A numerical example in Matsumura (2007) showed 3 estimations from different initial vectors, giving 3 different solutions with the same maximal likelihood. Plotting the impulse response functions, the macro factor affected the latent factor in a particular way in each one. Ang et al (2005) present a sub-identified specification, while other authors impose over-identifying restrictions, such as Dai and Philippon (2004), Hördahl et al (2002), Ang and Piazzesi (2003) and Amato and Luisi (2006). Over-identified models generally present sub-optimal solutions that distort the true relation between the factors.

The macro-finance model (MF) proposed by AP describes the evolution of the yield curve conditional on a state vector constituted of observed macroeconomic indicators and of latent factors, which have a vector autoregressive dynamics. In this representation it is possible to identify the interaction between exogenous macro shocks and monetary latent shocks, and evaluate the propagation of those shocks through time and maturities of the yield curve. Since it takes into account the joint behavior of the macro variables and the yield curve, unexpected macro fluctuations can be used to predict the yield curve, and the expected future rates can be used to predict macro variables.

Another article proposing an identification procedure is Pericoli and Taboga (2006). However, differently from the pure latent case, there is no canonical identification, and thus we tried to exhibit all possible options. One natural question to ask is if different identified specifications have different properties. We show that the likelihood is invariant under Dai-Singleton transformations, so that likelihood cannot be the only criterion to choose among the specifications. Furthermore, the response of the yield curve to unexpected state variable shocks also remains unchanged in invariant transformations. In fact, specification is only relevant for latent variables. In one point, though, the type of the specification could affect model properties: in the estimation, since some type of parameters may be harder to estimate, such as the premium.

Besides the MF model, we use the common factor model of the time series literature. It is similar to the MF model except that it ignores the no arbitrage restrictions among the rates. It is a descriptive model of the yield curve, in which we incorporate the same macro factors and the same number of latent factors. It does not depend upon underlying hypothesis for the risk premium. In spite of being less parsimonious, it is easier to estimate using the Monte Carlo Markov Chain (MCMC) inference approach, and the use of daily data supports a greater number of parameters.

The inclusion of macro variables is evaluated by comparing 3 specifications. The full model, a model excluding macro factors, and a model in which the macro factors are only included in the dynamics of the state variables.

Two markets are studied: the Brazilian domestic debt market issued in local currency and the sovereign bond market issued in dollars. The term structure of the former is measured using `DI x Pre` swaps, while of the latter uses Bloomberg zero-coupon data.

The Brazilian economy has a history of high inflation due to macroeconomic disequilibria which have been decreasing, but not completely vanished. This is reflected in the high levels of the interest rates. Since 1999, when a capital flow threat culminated in a forced devaluation of the currency, the government has adopted, in credible way, a monetary policy with inflation target and a floating exchange rate. The high interest rates have caused the singular characteristics of the market. In crises episodes, the monetary authorities rise the short rate, and the curve tend to show a horizontal or even decreasing profile.

In our context, the exchange rate and the expected inflation are relevant macroeconomic information which describe, jointly with the yield curve, the daily state of the economy. The interaction between the macro variables and the yield curve or the sovereign spreads and the propagation of the shocks and actions of the monetary authority constitute important questions for which we hope to use MF model as an instrument.

The occurrence of defaults in the past, the known vulnerability of the emerging markets to the international liquidity and the perception of the risk of the international lenders characterize the Brazilian sovereign market. To study it, we followed Duffie et al (2003), which utilized reduced credit risk model with affine interest rate for the Russian yield curve. Our model uses a discrete-time version of the Duffie and Singleton (1999) and Lando (1998) reduced models.

For the case of countries, reduced models have some advantages with respect to the structural models. The first credit risk models were proposed by Black and Scholes (1973) and Merton (1974). After Black and Cox (1976), default was modeled as the first time the stochastic process representing the assets of the firm crossed a default barrier. This is the structural approach, used in recent papers such as Leland and Toft (1996) in a model with endogenous default barrier. Deutsche Bank (2004) has structural model that incorporates a fiscal dynamics and is a step forward towards a more realistic model for countries. Moreira and Rocha (2003) proposes a 2-factor structural model for the Brazilian sovereign credit risk.

Differently from the structural models, in the reduced models the default event is an unpredictable stopping time, and do not depend on the difficult choice of the most adequate measure of indebtedness. We use a reduced model adapted to the discrete-time case and incorporating macro variables. This extension is used to analyze the effect of a measure of international liquidity, the US Treasury Bond yields, and a measure of volatility and risk aversion, the VIX, on the yield curve, and the proportion of the variance of the curve which can be attributed to those variables along the period posterior to the shock. The complement to that proportion is the effect of the idiosyncratic elements, such as the indebtedness

conditions of the country. The interaction and the effect of the propagation of the shocks along time and maturities will be analyzed.

The models are estimated using a Bayesian approach, the Monte Carlo Markov Chain (MCMC), which provides a sample of the posterior distribution of the parameters, of the prediction of the yield curve and of whatever transformations of those quantities. Differently from the classical case, the models are evaluated under a performance criterion in which the effect of the inherent uncertainty of the estimators is taken into account. This is possible in the Bayesian inference. We used the posterior predictive loss proposed by Gelfand and Ghosh (1998) and the information deviation criterion (DIC) proposed by Spiegelhalter (2002).

The objectives of the article are: 1) Propose and compare identified specifications of MF ; 2) Compare the MF and CF models to evaluate the adherence of the no arbitrage restrictions; 3) Evaluate the inclusion of macro variables.

In the following sections, we present the macro finance models, derive the no arbitrage restrictions, and empirically analyze the performance of the 2 models for the 2 sets of data. Then, dynamic properties of the macro shocks are analyzed in each market under different identification restrictions.

2 Macro-Finance Models

Following Ang and Piazzesi (2003), we derive the discrete-time pricing equations. Under the no arbitrage condition, the price at time t of an asset V_t that pays no dividend is

$$V_t = E^{\mathbb{Q}}[\exp(-r_t)V_{t+1}|\mathbb{F}_t], \tag{1}$$

with $\mathbb{Q} \sim \mathbb{P}$ being the martingale measure and $\mathbb{F}_t$ the filtration. We assumes that the short rate and the risk premium are affine functions of the state vector $X_t \in \mathbb{R}^p$, that is, $r_t = \delta_0 + \delta_1 X_t$ and $\lambda_t = \lambda_0 + \lambda_1 X_t$, where the dynamics of the state vector is a multifactor vector autoregression

$$X_t = \mu + \Phi X_{t-1} + \Sigma\epsilon_t. \tag{2}$$

Denote by ξ_t the Radon-Nikodym derivative $\frac{dQ}{dP} = \xi_t$. A discrete time "version" of the Girsanov theorem is assumed setting $\xi_{t+1} = \xi_t \exp(-\frac{1}{2}\lambda_t \cdot \lambda_t - \lambda_t\epsilon_{t+1})$, where $\{\epsilon_t\}$ are independent normal errors. Then, the Pricing Kernel will be $m_t = \exp(-r_t)\frac{\xi_{t+1}}{\xi_t}$, so that the price of a zero coupon bond maturing n+1 periods ahead is $p_t^{n+1} = E[m_{t+1}p_{t+1}^n]$. It can be proved by induction that the price of the bond will be exponential affine:

$$p_t^n = \exp(\alpha_n + \beta_n^\intercal X_t), \tag{3}$$

where:

$$\begin{gathered} \alpha_1 = -\delta_0,\ \beta_1 = -\delta_1, \\ \alpha_{n+1} = -\delta_0 + \alpha_n + \beta_n^\intercal(\mu - \Sigma\lambda_0) + \tfrac{1}{2}\beta_n^\intercal\Sigma\Sigma^\intercal\beta_n, \\ \beta_{n+1}^\intercal = -\delta_1^\intercal + \beta_n^\intercal(\Phi - \Sigma\lambda_1). \end{gathered} \tag{4}$$

Then $Y_t^n = -\log p_t^n/n = A_n + B_n X_t$, where $A_n = -\alpha_n/n$ and $B_n = -\beta_n/n$. Forming a vector of yields, we arrive at the same expression as in the continuous case,

$$Y_t = A + BX_t. \tag{5}$$

2.1 Adding Macro Factors

To incorporate macro factors, we extend the state vector of our economy to include observable macro variables M_t. Call the latent variables by θ_t. Then,

$$X_t = (M_t, \theta_t). \tag{6}$$

The short rate will be a combination of a Taylor Rule and an affine model:

$$r_t = \delta_0 + \delta_{11} \cdot M_t + \delta_{12} \cdot \theta_t. \tag{7}$$

The above specification allows the study of the inter-relations between macroeconomic questions, such as monetary policy, and finance problems, such as derivative pricing. Also, the affine tractability is completely retained. The same calculations as before lead to

$$Y(t, \tau) = A(\tau) + B^o(\tau) \cdot M_t + B^u(\tau) \cdot \theta_t. \tag{8}$$

2.2 Introducing the Spread

Credit risk can be incorporated into the model as follows. It is a discrete-time version of the reduced model of Lando (1998) and Duffie and Singleton (1999).

- Let h_s be the conditional probability at time s under a risk neutral probability $\mathbb{Q}$ of default between s and $s+1$ given the information available at time s in the event of no default by s.
- Let ρ_s and L_t be the recovery and loss rate upon default, respectively.
- Let V_t denote the price of a defaultable claim.

Then:

$$V_t = h_t e^{-r_t} E_t^{\mathbb{Q}}[\rho_{t+1}] + (1 - h_t) e^{-r_t} E_t^{\mathbb{Q}}[V_{t+1}]. \tag{9}$$

Using the Recovery of Market Value hypothesis, that is, $E_s^{\mathbb{Q}}[\rho_{s+1}] = (1 - L_s) E_s^{\mathbb{Q}}[V_{s+1}]$, it turns out that

$$V_t = h_t e^{-r_t} (1 - L_t) E_t^{\mathbb{Q}}[V_{t+1}] + (1 - h_t) e^{-r_t} E_t^{\mathbb{Q}}[V_{t+1}], \tag{10}$$

or

$$V_t = E_t^{\mathbb{Q}}[V_{t+1}](h_t e^{-r_t}(1 - L_t) + (1 - h_t) e^{-r_t}) = E_t^{\mathbb{Q}}[V_{t+1}] e^{-r_t}(1 - h_t L_t). \tag{11}$$

Now, note that $\exp(-h_t L_t) \approx 1 - h_t L_t$. If we set that relation to be an equality (just redefine L_t), we will have:

$$V_t = E_t^{\mathbb{Q}}[V_{t+1}] \exp(r_t + h_t L_t). \tag{12}$$

as in the continuous case. Call the additional term $s_t = h_t L_t$ the spread due to default. The spread s_t will be another state variable.

The price d_t^n of a defaultable bond is

$$d_t^{n+1} = E_t^{\mathbb{Q}}[\exp(-r_t - s_t) d_{t+1}^n]. \tag{13}$$

The solution to this problem is similar to the previous one:

$$d_t^n = \exp(\alpha_n + \beta_n X_t), \tag{14}$$

where:

$$\begin{gathered} r_t = \delta_0^r + \delta_1^r X_t, \;\; s_t = \delta_0^s + \delta_1^s X_t, \\ \alpha_1 = -\delta_0^r - \delta_0^s, \;\; \beta_1 = -\delta_1^r - \delta_1^s, \\ \alpha_{n+1} = -\delta_0^r - \delta_0^s + \alpha_n + (\mu^\intercal - \lambda_0^\intercal \Sigma)\beta_n + \tfrac{1}{2}\beta_n^\intercal \Sigma^\intercal \Sigma \beta_n, \\ \beta_{n+1} = -\delta_1^r - \delta_1^s + (\Phi - \lambda_1^\intercal \Sigma)\beta_n. \end{gathered} \tag{15}$$

Then, fixing again $Y_t^n = -\log p_t^n / n = A_n + B_n X_t$, where $A_n = -\alpha_n / n$ and $B_n = -\beta_n / n$ and piling a vector of yields,

$$Y_t = A + B X_t. \tag{16}$$

as before.

In this text, r_t will be an observable state vector, the US Treasury 1-month yield, so that $\delta_0^r = 0$ and $\delta_1^r = (1, 0, .., 0)$ when r_t is the first variable.

2.3 Default Probabilities

The term structure of default probabilities is given by

$$\Pr(t, \tau) = E^{\mathbb{P}}\left[\exp\left(-\int_t^{t+\tau} s_t dt\right) | \mathbb{F}_t\right]. \tag{17}$$

It turns out that

$$\Pr(t, \tau) = \exp(\alpha^{\Pr}(\tau) + \beta^{\Pr}(\tau) \cdot X_t), \tag{18}$$

with $\alpha^{\Pr}$ and $\beta^{\Pr}$ given by solutions of recursive equations:

$$\begin{gathered} R_t = \delta_0^R + \delta_1^R X_t, \;\; s_t = \delta_0^R + (\delta_1^R - (1, 0, .., 0)) X_t, \\ \alpha_1^{\Pr} = -\delta_0^R, \;\; \beta_1^{\Pr} = -\delta_1^R + (1, 0, .., 0), \\ \alpha_{n+1}^{\Pr} = -\delta_0^R + \alpha_n^{\Pr} + \mu^\intercal \beta_n^{\Pr} + \tfrac{1}{2}\beta_n^{\Pr\,\intercal} \Sigma^\intercal \Sigma \beta_n^{\Pr}, \\ \beta_{n+1}^{\Pr} = -\delta_1^R + (1, 0, .., 0) + \Phi \beta_n^{\Pr}. \end{gathered} \tag{19}$$

Here $R_t = r_t + s_t$ is the short rate of the Brazilian sovereign yield curve. Note that the objective measure is used. Thus, the log of the probabilities is again an affine function of the state variables,

$$\log \Pr(t, \tau) = \alpha^{\Pr}(\tau) + \beta^{\Pr}(\tau) \cdot X(t). \tag{20}$$

3 Identification

The above affine term structure model must be identified, meaning that the parameter space, constituted of vectors $\Psi = (\delta_0, \delta_1, \Phi, \mu, \lambda_0, \lambda_1, \Sigma)$, must be constrained. We follow DS and find invariant transformations that preserve the short rate and the premium process, so that all yields are preserved. Since many choices of parameters lead to the same observed yields, clearly one has to make a choice. In the following, we present a proposition showing necessary conditions. We could not find sufficient conditions, for we do could not prove that there are not additional rotations.

By contrast, the alternative approach of Collin-Dufresne et al (CGJ, 2006) uses "rotations" that lead to observable state variables. By using the observed yields and linear equations relating them to the state variables, they manage to globally identify one set of risk-neutral parameters.

On the other hand, one could define identification in terms of the likelihood. In Hamilton (1994) it can be found a definition in which a model is globally identified for a particular parameter vector Ψ_0 if for any Ψ there exists a possible realization of the observable data for which the value of the likelihood at Ψ is different from the value of the likelihood at Ψ_0. Neither of the above approaches guarantee this more strict definition. Indeed, the possibility of the existence of transformations other than DS are not precluded. As for the CGJ approach, although risk-neutral parameters are proved to be identified, the rest of the parameters (the $\mathbb{P}$-drift of the state process) could in principle still be sub-identified.

Estimating a sub-identified model does not necessarily produce wrong results. Certainly, the dimension of the problem is higher, but the same maximal mean likelihood value would be achieved, taking numeral problems (global maximization) aside. But confidence intervals and economic interpretation of unidentified parameters and latent factors, such as when considering the effect of monetary shocks on the yield curve, would be meaningless. On the other hand, estimating a over-identified specification, in which the additional restrictions do not come from economic restrictions, will result in a sub-optimal solution.

We observe that both approaches agree in number of restrictions for the Gaussian case. We thus define that a specification is *idenfied* if all DS degrees of freedom of the parameter space are spent.

There are two main types of transformations of the parameters:

- Transformation L is defined by a nonsingular matrix L and a vector ν such that:
$$L = \begin{pmatrix} \mathbf{I} & 0 \\ \alpha & \beta \end{pmatrix}, \ \nu = \begin{pmatrix} 0 \\ \nu^{\theta} \end{pmatrix}, \tag{21}$$
$$T_L(\Psi) = (\delta_0 - \delta_1^{\intercal} L^{-1}\nu, (L^{\intercal})^{-1}\delta_1, L\Phi L^{-1}, \nu + L\mu, \lambda_0 - \lambda_1 L^{-1}\nu, \lambda_1 L^{-1}, L\Sigma). \tag{22}$$
It is the most general invariant transformation affecting the state vector. Note that L preserves the macro factors but changes the latent factors.

Indeed, it will be used to identify the latend factors. Bellow we will also use particular variants of the operator when $\alpha = 0$ and the square matrix β is diagonal ($\mathbf{D}$), triangular ($\mathbf{T}$) or a rotation ($\mathbf{S}$).

- Transformation O uses a rotation matrix O, which takes a vector of unobserved, independent Brownian motions into another vector of independent Brownian motions:

$$T_O(\Psi) = (\delta_0, \delta_1, \Phi, \mu, O\lambda_0, O\lambda_1, \Sigma O^\intercal). \tag{23}$$

  The rotations do not affect the state factors and is used solely to identify the order of exogeneity of the shocks. Using this operator, we impose that Σ is a lower triangular matrix, which is a usual VAR model restriction. Our choice of ordering is that macro factors do not react contemporaneously to monetary policy.

Thus, we use the number of free entries in α, β, ν and O to restrict Ψ and give the minimal conditions to identify the specification. All specifications will use them to put zeroes in the parameter space. For example, we can impose $\mu^\theta = 0$ using ν^θ.

We divide the identifications in 2 types. The first type solves linear equations to set parameters of Σ to zero, and the second type need an additional hypothesis to simplify non-linear restrictions on Φ or $\Phi^\star$.

Proposition 1 *1) The affine model with observable factors with one of the restrictions bellow is identified in DS sense. 2) Under the hypothesis that Φ can be decomposed as $P\Lambda P^{-1}$, where Λ is a real diagonal matrix, the affine model with observable factors with one of the restrictions bellow is identified in DS sense.*

Proof. 1) The identification is achieved by spending most of the restrictions on Σ or λ_1.

The matrices α and β are chosen such that:

$$L\Sigma = \begin{pmatrix} \mathbf{I} & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \Sigma_{MM} & 0 \\ \Sigma_{\theta M} & \Sigma_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \Sigma_{MM} & 0 \\ 0 & \mathbf{I} \end{pmatrix} \tag{24}$$

Then, apply the rotation transformation S such that $\Phi_{\theta\theta}$ becomes lower triangular. The operator S commutes with $L\Sigma$ so that it will only rotate orthogonal Brownian motions and will not affect the other parameters, resulting in

$$\Sigma = \begin{pmatrix} \Sigma_{MM} & 0 \\ 0 & I \end{pmatrix}, \quad \mu = 0, \quad \Phi = \begin{pmatrix} \Phi_{MM} & \Phi_{M\theta} \\ \Phi_{\theta M} & \Phi_{\theta\theta} \end{pmatrix}, \tag{25}$$

where $\Phi_{\theta\theta}$ is lower triangular. This completes one identification. Alternatively, we can impose a lower triangular $\Phi^\star_{\theta\theta}$ instead of $\Phi_{\theta\theta}$. Another option is obtained by choosing β such that $\Sigma_{\theta\theta}$ is diagonal and then apply the diagonal transformation D such that $\delta_\theta = 1$:

$$(D^{-1})^\intercal \delta = \begin{pmatrix} I & 0 \\ 0 & \widetilde{D}^{-1} \end{pmatrix} \begin{pmatrix} \delta_M \\ \delta_\theta \end{pmatrix} = \begin{pmatrix} \delta_M \\ 1 \end{pmatrix}. \tag{26}$$

The scale of the variance of the latent factors is equally adjusted if either $\Sigma_{\theta\theta} = I$ or $\Sigma_{\theta\theta}$ is diagonal and $\delta_\theta = 1$.

Finally, note that we can impose restrictions on λ_1 instead of Σ. For example,

$$\lambda_1 L^{-1} = \begin{pmatrix} \lambda_1^{MM} & \lambda_1^{M\theta} \\ \lambda_1^{\theta M} & \lambda_1^{\theta\theta} \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ -\beta^{-1}\alpha & \beta^{-1} \end{pmatrix} = \begin{pmatrix} \lambda_1^{MM} & \lambda_1^{M\theta} \\ 0 & \mathbf{I} \end{pmatrix}, \quad (27)$$

so that we elimate $\lambda_1^{\theta M}$ and $\lambda_1^{\theta\theta}$. The same is true with respect of λ_0 and μ.

2) To find more identifications, this time restricting Φ, systems of nonlinear equations would have to be solved. To avoid this, we assume that Φ can be decomposed as $P\Lambda P^{-1}$, where Λ is a real diagonal matrix. If Φ has real and distinct eigenvalues, this is always possible.

Since $\Phi = P\Lambda P^{-1}$, then $L\Phi L^{-1} = LP\Lambda(LP)^{-1}$ and A can be chosen such that

$$LP = \begin{pmatrix} \mathbf{I} & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} P_{MM} & P_{M\theta} \\ P_{\theta M} & P_{\theta\theta} \end{pmatrix} = \begin{pmatrix} P_{MM} & P_{M\theta} \\ 0 & P_{\theta\theta} \end{pmatrix}. \quad (28)$$

Then $(LP)^{-1}$ and the transformed $\widehat{\Phi} = L\Phi L^{-1}$ will also be upper block triangular. That is, $\widehat{\Phi}_{\theta M} = 0$.

The first identification of this type is the following. Choose β such that $\beta P_{\theta\theta}$ is a diagonal matrix, then $\widehat{\Phi}_{\theta\theta}$ will also be diagonal. Next, use the scale transformation such that $\delta_\theta = 1$, completing the identification.

Alternatively, assume that $\Phi^\star = P\Lambda^\star P^{-1}$ and repeat the above identification using $\Phi^\star$. Another possibility is to impose a lower triangular $\beta P_{\theta\theta}$, which implies in a lower triangular $\widehat{\Phi}_{\theta\theta}$. Then use the triangular transformation T with a lower triangular matrix that impose a diagonal $\Sigma_{\theta\theta}$. Finally, the identification finishes by choosing a diagonal transformation D such that $\delta_\theta = 1$. Note that each transformation alters all the parameters, but choosing in the correct order, they preserve the desired properties. For instance, T and D will not alter the lower triangular condition of $\widehat{\Phi}_{\theta\theta}$ and $\widehat{\Phi^\star}_{\theta M} = 0$ as before. Table 1 shows the alternatives.

Table 1. Summary of Identifications.

	$\Phi_{\theta M}$	$\Phi_{\theta\theta}$	$\Phi^\star_{\theta M}$	$\Phi^\star_{\theta\theta}$	$\Sigma_{\theta M}$	$\Sigma_{\theta\theta}$	δ^θ	λ_1
Type1	Full	Full	Full	L Tr	0	I	Full	Full
Type1	Full	Full	Full	L Tr	0	Diag	1	Full
Type1	Full	L Tr	Full	Full	0	I	Full	Full
Type1	Full	L Tr	Full	Full	0	Diag	1	Full
Type1	Full	Full	Restr	Restr	Full	Full	Full	Restr ■
Type1	Restr	Restr	Full	Full	Full	Full	Full	Restr
Type2	Full	Full	0	Diag	Full	L Tr	1	Full
Type2	Full	Full	0	L Tr	Full	Diag	1	Full
Type2	0	Diag	Full	Full	Full	L Tr	1	Full
Type2	0	L Tr	Full	Full	Full	Diag	1	Full

Next, we prove that the identifications have the same likelihood, pricing equations and impulse response for Y.

Proposition 2 *DS transformations preserve the pricing equation and the impulse response function for Y.*

Proof. See Appendix. ■

Proposition 3 *If $\widetilde{m}_0 = Lm_0$ and $\widetilde{C}_0 = LC_0L^\intercal$, then DS transformations preserve the likelihood of the affine model with observable factors under the Kalman Filter.*

Proof. See Appendix. ■

As a matter of fact, we also show that:

Proposition 4 *DS transformations preserve the likelihood of the affine model with observable factors under Chen-Scott.*

Proof. See Appendix. ■

A restriction on Φ result in changes in dynamic properties, while a restriction on Σ affects one-period shocks.

It turns out that any specification, if correctly estimated, lead to the same model properties. However, it can be numerically simpler to estimate a model with less premium parameters, which are highly nonlinear and sampled using Metropolis-Hasting instead of Gibbs sampling, which has better convergence properties.

4 Inference

The last section showed that the models only differ in the specification of the matrices A, B which relates the yield curve to the latent factors. In general, the model can be defined as

$$Y_t = A(\Psi) + B(\Psi)\theta_t + \sigma e_t,\ e_t \sim N(0, I_n) \tag{29}$$

$$\theta_t = \mu + \phi\theta_{t-1} + \Sigma u_t,\ u_t \sim N(0, I_p) \tag{30}$$

where $\Psi = (\mu, \phi, \sigma, \zeta, \theta)$ and the definition of ζ depend on the model and is summarized below.

CF	NA
$\zeta = (A, B)$	$\zeta = (\delta_0, \delta_1, \mu^*, \phi^*, \Sigma)$

The likelihood $L(\Psi) = p(Y|\Psi) = p(Y|\theta, \Psi)p(\theta|\Psi)p(\Psi)$, where we assume a non informative prior $p(\Psi) = 1$, and

$$p(Y|\theta, \Psi) = \prod_t p(Y_t|\theta_t, \psi) = -1/2\left[T\sum_i \log(\sigma_i^2) + \sum_t \sum_i (u_{it}^2/\sigma_i^2)\right], \tag{31}$$

$$p(\theta|\Psi) = \prod_t p(\theta_t|\theta_{t-1},\psi) = -1/2\left[T\sum_i \log(|\Sigma|) + \sum_t (e_t^\intercal(\Sigma^\intercal\Sigma)^{-1}e_t\right], \quad (32)$$

$$u_{it} = Y_{it} - A_i(\delta_0,\Sigma,\mu^*,\phi^*) - B_i(\delta_1,\phi^*)X_t, \quad e_t = \theta_t - \mu - \phi\theta_{t-1} \quad (33)$$

The distribution of the parameters,

$$p(\theta|Y,M_t,\psi) \propto p(Y|\theta,M_t,\Psi)p(\theta|M_t,\Psi)p(\Psi), \quad (34)$$

cannot be derived analytically, but the Clifford-Hammersley theorem guarantees that the recursive sampling of subsets of parameters, obtained from the complete conditional distributions, converges to the joint distribution. The subsets are chosen in a convenient way such that the subproblems have, when possible, analytical solutions and known complete conditional distributions, as is the case of subproblems 1-3 bellow. These problems correspond to, respectively, an estimation of a VAR model, the variance of known random variables, and the extraction of unobservable factors from a multivariate dynamic model. The distributions calculated in each step of the algorithm are:

1. $(\mu^w,\phi^w) \sim p(\mu,\phi|\sigma^w,\zeta^w,\theta^w)$,
2. $\sigma^w \sim p(\sigma|\mu^w,\phi^w,\zeta^w,\theta^w)$,
3. $\theta^w \sim p(\theta|\mu^w,\phi^w,\zeta^w,\sigma^w)$,
4. $\zeta_i^w \sim p(\zeta_i|\zeta_{-i}^w,\mu,\phi,\Sigma,\sigma,\theta)$,

We have:

Subproblem1: $p(\mu,\phi|\sigma^w,\zeta^w,\theta^w) \sim N((X^\intercal X)^{-1}X^\intercal X^*,(X^\intercal X)^{-1}\otimes\Sigma)$, where $X = (\theta_1^w,...,\theta_{T-1}^w)^\intercal$, $X^* = (\theta_2^w,...,\theta_T^w)^\intercal$.

Subproblem2: $p(\sigma|\mu,\phi,\zeta,\theta) \sim \mathcal{IG}(diag(e^\intercal e))$, where $e = Y - A - BX$, and $\mathcal{IG}$ is the inverse gamma distribution.

Subproblem3: $p(\theta|\mu,\phi,\sigma,\zeta) = \prod_t p(\theta_t|\mu,\phi,\sigma,\zeta)$, where $p(\theta_t|\mu,\phi,\sigma,\zeta) = p(\theta_t|D_T) \sim N(h_t,H_t)$ is the FFBS algorithm defined in the Appendix.

The subproblems 1-3 are common to all models. However, subproblem 4 depends on the definition of ζ. In the case of the CF model, $\zeta = (A,B)$ is estimated without restrictions. Subproblem 4 becomes

$$(\zeta|\mu,\phi,\sigma,\theta) = (A,B|\mu,\phi,\sigma,\theta) = N((\theta^{w\intercal}\theta^w)^{-1}\theta^{w\intercal}Y,(\theta^{w\intercal}\theta^w)^{-1}\otimes\sigma^2). \quad (35)$$

In the model NA, the parameter ζ do not have known conditional distribution, and its distribution will be obtained through the Metropolis-Hastings rejection method (Gamerman, 2001, and Johannes and Polson, 2003). The proposal is sampled from a normal distribution centered on the value of the previous iteration, with arbitrarily fixed variance such that the acceptance ratio lies in the interval $[0.3,0.8]$: $p(\zeta_i|\zeta_{i-1},\mu,\phi,\sigma,\theta) \sim N(\xi_i^k,c)$ and accepts if $p(Y|\xi_i^k) - p(\theta|\xi_i^{k-1}) > u$, $u \sim U(0,1)$.

4.1 Performance Criteria

The models under investigation have a different number of parameters, and hence they must be compared emphasizing forecasting performance or adherence to data. Gelfand and Ghosh (1998) proposed the minimum posterior predictive loss (PPL) criterion emphasizing forecasting performance. Spiegelhalter (2002) proposed the DIC criterion emphasizing adherence. Besides those measures, we will calculate Theil's U statistical measure, which consists of normalizing the MSE of out-of-sample forecasts and of in-sample adherence with respect to corresponding measures using random walks.

4.1.1 Posterior predictive loss (PPL)

For each point of the distribution of the estimators $\Psi^w \sim (\Psi|Y)$ there corresponds a forecasting for the yield curve $Y|\ \Psi^w$. Gelfand and Ghosh (1998) proposes a loss function penalizing the expected error $E(Y|\Psi^w) - Y$ and the variance of the forecasts $Y|\Psi^w - E(Y|\Psi^w)$. In our case, the target variable is multivariate, so that we take the mean of the expected losses calculated for each of the maturities. In other words, the criterion is:

$$\text{PPL} = \sum_i \sum_t \left(Y_t^i - E(Y_t^i|\Omega)\right)^2 + \frac{1}{2} \sum_i \sum_t \frac{1}{N_w} \sum_w \left(E(Y_t^i|\Psi^w) - E(Y_t^i|\Omega)\right)^2, \tag{36}$$

4.1.2 Divergence of Information Criterion (DIC)

Spiegelhalter (2002) proposed a generalization of the AIC criterion based on the distribution of the divergence $D(\Psi) = -2 \log L(\Psi)$:

$$\text{DIC} = E(D(\Psi)) - p_d = 2E(D(\Psi)) - D(E(\Psi)), \tag{37}$$

where $p_d = E(D(\Psi)) - D(E(\Psi))$ measures the equivalent number of parameters in the model, $E(D(\Psi))$ is the mean of the divergences taken in the posterior distribution of the estimators and $D(E(\Psi))$ is the divergence calculated at the mean point of the posterior distribution of the estimators.

Banerjee et al (2004) claims that LLP and DIC evaluate the fitting and penalize the degree of complexity of the models, but that the DIC takes into account the likelihood on the space of the parameters and PPL on the predictive space. Thus, when the main interest lies is forecasting, the PPL is to be preferred, whereas when the capacity of the model to explain the data is more interesting, DIC should be used.

5 Results

In this exercise, the interaction between the yield curve and macro variables in two markets having different characteristics and relevant questions is analyzed.

In the first market, the domestic curve of public debt, the question is evaluate the interaction of the yield curve and the exchange rate and the expected inflation. In the second market, constituted of sovereign bonds traded in the international market, the question is to evaluate the importance of the international conditions on the yield curve. In the latter case, we consider that Brazil is a small economy, in which the international macro variables are not affected by the Brazilian yield curve. The decomposition of the correlation of the path of the yields of 9 maturities indicates that, for both markets, 2 stochastic components explain more than 95% of the total variance, which suggests that 2 latent factors suffice.

The dynamics of the curve is described by state variables following a VAR. As usual in the structural VAR literature, the structural shocks on the state variables were identified supposing independence, and with this hypothesis, we calculate the dynamic effect of the shocks on the state variables and yield curve.

The model has two time dimensions, the unity of time of the frequency of observations of the sample, and the unity of measure of the maturity. Also, the impulse response measures the effect of the shocks according to the relative size of the lag with respect to the frequency of observations. The effect of the unexpected rise of the exchange rate on each of the macro factors and on the short and long rates certainly varies with the size of the lag.

In the VAR literature, the usual way to identify the structural shocks is choosing an ordering from more exogenous to more endogenous shocks. Since this is arbitrary and determines dynamic properties of the model, the identification of the VAR models is a polemic question. Our model requires those identifications too, and we suppose that unexpected alterations of the financial market react contemporaneously to economic innovations, but not the contrary. In other words, we admit that the turbulences of the financial market propagate to the economic variables, but not contemporaneously to the financial shock. This hypothesis seems reasonable for the domestic market, and is used by other authors such as Ang and Piazzesi (2003). The same is assumed for the external market, for stronger reasons. In fact, one does not expect that the Brazilian bond market should alter international economic variables such as the FED Fund or VIX, so here the macro factors are not affected by the Brazilian financial factors, contemporaneously or through time.

The dynamics of the model still depends on identification of latent factors in addition to the previous identification of VAR models. However, the impulse response function of the yield curve does not depend on the identification of the latent factors XXX under the hypothesis that the shocks on the observed variables are more exogeneousXXX. Thus, the proportion of the variance of the forecasting due to macro factors also does not depend on the identification. This quantity measure the degree of interconnection between the economic factors and the yield curve, that is, how much one market affects the other one.

The Brazilian economy have been following a process of gradual adjustment and of maturing of the financial market since 1994. In the beginning of 1994, Brazil concluded the renegotiation of the 1985 default, and, in the end of 1994, a stabilization plan was implemented which reduced the high inflation levels. In

1999, after a speculative attack on the Central Bank reserves resulting in a forced devaluation of the local currency, the monetary regime changed to free floating combined with inflation target. Since the model assumes that the parameters are stable, it is recommended that the estimation is done using data posterior to those events, which considerably reduces the length of the sample. The size of the time interval of the data, the volatility of the market and the availability of the information motivated the specification of the model with daily frequency.

In the models with daily frequency, the transition is normally defined with a lag of one day, and the likelihood is defined for forecasts of the next day. This is a too short horizon for analyzing the dynamics and linkage among variables. So, the likelihood was altered to consider 1-month forecasts. Since 1 month corresponds to 21 commercial days on average, this is equivalent to altering the lag of the transition equation to 21 days. In fact, other lags were tested, but 21 days presented the best results.

The following subsections present the results for each market. The comparison between the CF and MF models evaluates the degree of restriction that the no arbitrage condition impose. For each case, we considered specifications which evaluate wether: 1) the macro variables aggregate information for the forecasting of the yield curve; 2) more than one latent factor is needed; 3) the choice of the identification affects the inference; 4) the hypothesis that the macro factors only affect the yield curve through the latent factors is supported by the data.

The inference was implemented using MCMC using 6 independent chains with 3500 replications, of which the last 1000 iterations were considered for obtaining the out-of-sample estimates. In all cases using the Metropolis-Hasting algorithm, the deviance of the normal distribution of the sampling was adjusted such that the rate of acceptance of the parameters estimated with the rejection method remained in the interval [0.3, 0.8].

5.1 Domestic Market

The Brazilian economy is well known for its high level of interest rates. At events of liquidity tightness, the yield curve often exhibits a singular decreasing shape. Among possible factors explaining it, we list the fiscal vulnerability, the volatility of the market and the fact that a too high interest rate is not sustainable in the long run.

In the period under study, the exchange rate regime is floating, and hence the innovations of the exchange market constitute an important factor for the formation of the price of imported products. Likewise, agent's expectations about the consumer price index influence the prices of domestic goods and services. The expected inflation can be measured in may ways, of which we consider two. The first is provided by the Central Bank, which produces a formal survey among agents of the financial markets about the prediction of many variables including the consumer prices - the FOCUS research. The other source comes from swap contracts traded in the BM&F, the $INPC\ x\ DI$ swap, which yields the difference between the inflation rate measured by the consumer's price index ($INPC$) and the floating interest rate observed for the contracted

maturity. The difference between the rates for the same maturity is a measure of the expected inflation for a horizon equal to the maturity. This estimate contains a premium risk that was supposed constant and discarded. The 9-month ahead expected inflation was selected. The interval of time that was used in the estimation was [04/2002, 10/2005]. The exchange rate is the mean of buy rates provided by *Ipeadata*.

A contingent contract named *DI x Pré* Swap is traded in the BM&F in which the seller is committed to pay the accumulated interest of the short rate *DI* observed during the term of the contract. The prices of this swap for different maturities provide a measure of the yield curve. One analysis of this curve indicates that 99% of the variance of the yield vector is described by 2 canonical components, which motivated the definition of a specification with 2 latent factors.

The option for a daily sample limits the study to the analysis of nominal shocks, the only available at this frequency. A preliminary analysis suggested that the interest rates, expected inflation and the exchange rate are the 3 main prices of the economy and sources of nominal shocks.

In the inflation target regime, the Central Bank reacts to nominal shocks, fixing the economy's basic short interest rate - the Selic - at the periodic Monetary Policy Committee (COPOM) meetings. Our model is fed with another short rate, the 1-month swap rate, which continuously floats around the Selic path. Hence, in our model the short rate response to the macro shocks can be interpreted as the effect of the systematic reaction of the monetary policy plus the non-systematic market driven fluctuations. The shocks on the latent factors represent all the other sources of information. The proposed models will quantify the relative importance of this two sets of sources of innovations for the path of the economic variables and the evolution of the yield curve. In addition to the MF versions, corresponding CF versions were estimated.

Different versions were specified to empirically answer questions about identification. Starting from a reference specification with 2 latent factors and 2 macro variables and identified with restrictions on the premium (labeled $\Phi_2^\star$), we considered versions: 1) excluding the macro variables (x), 2) excluding one factor ($\Phi_1^\star$); 3) identifying restricting the covariance matrix (Σ_2), 4) over-identified with the hypothesis that the macro variables do not enter the observation equation ($\delta\Phi_2^\star$). The tables 2A and 2B present the results of the performance criterion for the various versions and for the two inflation measures.

Tab 2A
Comparing Models: Deviance of Information Criterion

	Latent	Latent +swap			
	x	$\Phi_1^\star$	Σ_2	$\Phi_2^\star$	$\delta\Phi_2^\star$
MF	-3.451	-2.758	-2.990	$\mathbf{-3.516}$	-3.437
CF	-3.568	-3.051	$\mathbf{-3.788}$		-3.571
		Latent + Focus			
MF		-2.911	-3.321	$\mathbf{-3.499}$	-3.442
CF		-3.163	-3.565		$\mathbf{-3.578}$

Tab 2B
Comparing Models: PPL Criterion

	Latent	Latent +swap			
	x	$\Phi_1^\star$	Σ_2	$\Phi_2^\star$	$\delta\Phi_2^\star$
MF	0.064	0.220	0.068	**0.050**	0.069
CF	0.049	0.137	**0.041**		0.046
		Latent + Focus			
MF		0.135	**0.042**	**0.042**	0.060
CF		0.103	**0.040**		0.046

Table 2 shows that under the two information criterion: 1) the CF model presents better results than MF; 2) the specification with the best performance is the exactly identified version with restrictions over the premium $\Phi_2^\star$; 3) the tested over-identifying restrictions $\delta\Phi_2^\star$ were rejected by the data; 4) the identification Σ_2 is equal or worse than the $\Phi_2^\star$; 5) the choice of the indicator of expected inflation is ambiguous: for the DIC criterion the best measure is the *INPC x DI* swap, and for the Gelfand and Gosh PPL the best indicator is produced by the Focus Survey.

Using selected versions of the MF model for the two measures of expected inflation, impulse response functions and variance decompositions were calculated for short and long horizons (1m and 18m after the shock). Table 3 presents the proportion of the variance that is explained by the economic shocks for selected maturities of the yield curve (1,9,36)-months - and for the economic variables itself in the long run.

Tab 3
Percentage of the variance explained by economic shocks

	Short Run			Long Run				
Maturity	1m	9m	36m	1m	9m	36m	EX	Infl
Swap-$\Phi_2^\star$	0.64	0.73	0.76	0.71	0.71	0.71	0.74	0.71
Swap-Σ_2	0.59	0.59	0.64	0.94	0.94	0.89	0.88	0.9
Focus-$\Phi_2^\star$	0.21	0.39	0.46	0.34	0.36	0.51	0.88	0.81
Focus-Σ_2	0.02	0.01	0.01	0.93	0.84	0.65	0.71	0.73

Table 3 shows the proportion of the variance explained by macro shocks, a result that should not depend upon the identification of the latent factor. In the case of the MF model, the economic shocks explain 30-70% of the floating of the yield curve, and about 80% of the floating of the economic variables. In the case

of the CF models the results also show the importance of the economic shocks in the long run, even though the results on the short run are less consistent.

The response of the yields to the 2 nominal shocks is similar, and the yield of greatest maturity reacts with greater intensity than the short maturity. This result shows that the market is more sensitive to the shocks than the monetary authorities itself.

Table 4 evaluates the in-sample fitting of the $\Phi_2^\star$ specification. It shows the measurement errors of the yields of the short, medium and long-term rates.

Tab 4

Measurement errors for selected maturities. Domestic yields.

	x	$\Phi_1^\star$	Σ_2	$\Phi_2^\star$	$\delta\Phi_2^\star$
			BM&F		
1m	0.0026	0.0286	0.0032	0.0052	0.0024
9m	0.0024	0.0054	0.002	0.0056	0.0023
36m	0.0206	0.0151	0.0083	0.0156	0.0198
			Focus		
1m	0.0026	0.0218	0.003	0.003	0.0026
9m	0.0024	0.0085	0.0035	0.0041	0.0022
36m	0.0206	0.0098	0.0072	0.0087	0.0172

5.2 Sovereign Bonds Market

To study an emerging market yield curve, it is important to model the credit risk component. Three types of models are currently the most used: the structural, the reduced, and the econometric models. Some recent developments in the structural models addressed the sovereign case, but at a cost of a greater complexity (see Guezzi and Xu, 2002). Econometric modelling have been used in international comparative studies (see García-Herrero and Ortiz, 2005), but it does not fit the no-arbitrage framework of the macro-finance models. Therefore, we use reduced models, as Duffie et al (2003), which analyses the sovereign Russian bonds, combined with Ang and Piazzesi (2003) affine model. To make the model as simple as possible, the short rate is assumed observable - it is the US Treasury 1-month rate, and provided as a zero-coupon constant maturity data by the FED, and part of the state vector. Shocks of the FED and VIX represent the effect of the international conditions and the latent factor shocks represent the idiosyncratic conditions in Brazil, for example the effect of alterations of the domestic fundamentals, or the agent's expectation about the future evolution of the fundamental.

We used a constant maturity zero-coupon Brazilian sovereign yield curve that was calculated by Bloomberg. The sample consists of daily data from 01/1999 to 09/2005, composed of maturities {1,6,12,24,36,60,84,120,240}-months. The estimation was conducted using the same specifications as in the case of the domestic curve, except the over-identified version.

The use of external instead of domestic macro factors is discussed in Matsumura (2006), which documents that domestic factors did not represent an

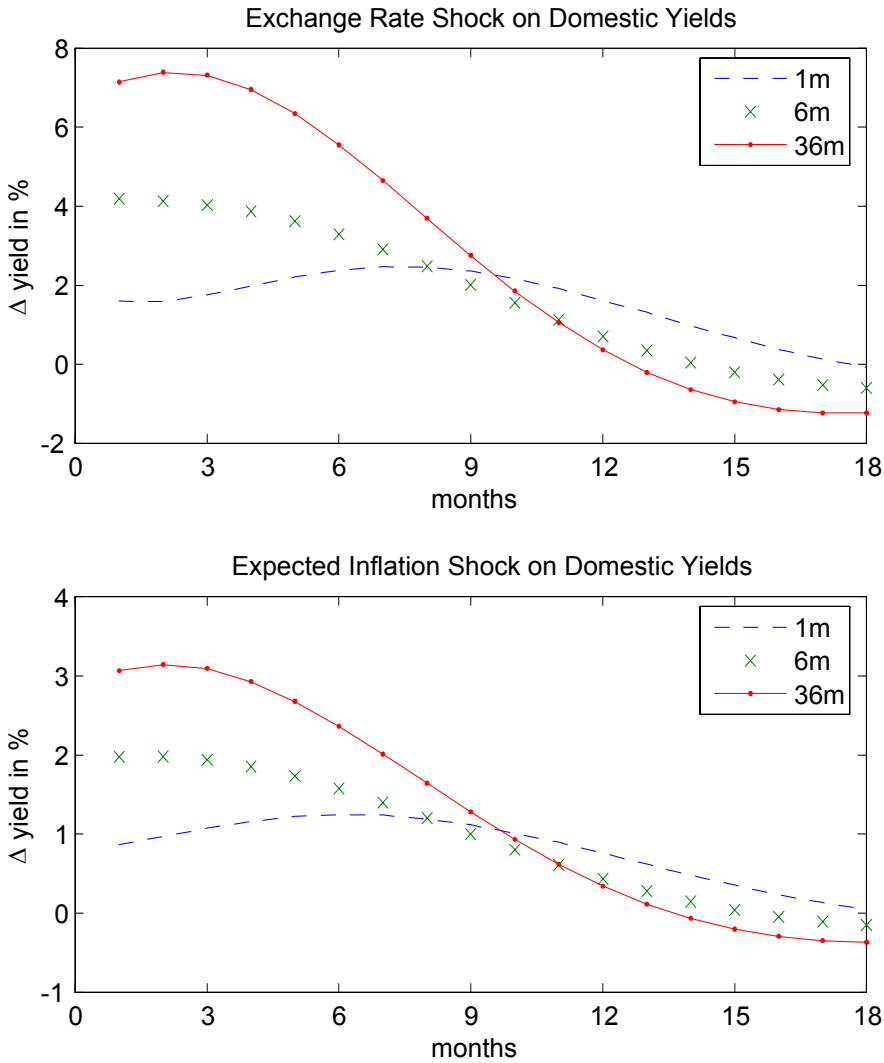

Figure 1: Effect of domestic macro shocks on the domestic term structure.

expressive source of information for the Brazilian sovereign yields.

Tab 5

Performance criterion: Brazilian sovereign yields.

	x	$\Phi_1^\star$	Σ_2	$\Phi_2^\star$	$\delta\Phi_2^\star$
			DIC		
MF	-5.537	-5.721	-5.370	$\mathbf{-5.904}$	-5.220
CF	-5.367	-5.800	-5.812		$\mathbf{-6.400}$
			PPL		
MF	0.502	0.143	0.214	**0.123**	0.151
CF	0.144	0.108	**0.074**		0.107

The version that presents the best performance under both criterion is the one having external variables, 2 factors, and identified with restrictions in the matrix $\Phi^\star$. This means that the Fed Fund and VIX volatility aggregate information. Moreover, the second latent factor added information and the CF model presented the best performance.

Tab 6

Percentage of the sovereign rates forecasts explained by external shocks.

	Short run			Long run		
Maturity	1m	3y	20y	1m	3y	20y
			sovereign rates			
$\Phi_2^\star$	0.20	0.13	0.22	0.21	0.15	0.33
Σ_2	0.12	0.05	0.39	0.09	0.15	0.76
$\Phi_1^\star$	0.19	0.40	0.74	0.30	0.50	0.82
$\delta\Phi_2^\star$	0.12	0.11	0.36	0.22	0.10	0.53
			log of the survival prob			
$\Phi_2^\star$	0.43	0.78	0.91	0.64	0.87	0.94
Σ_2	0.72	0.96	0.97	0.86	0.98	0.99
$\Phi_1^\star$	0.99	1.00	1.00	1.00	1.00	1.00
$\delta\Phi_2^\star$	0.55	0.81	0.90	0.69	0.88	0.94

Table 6 shows the proportion of the variance of the spread forecasts that is explained by external shocks. The results for the short rate are similar and vary between 10% to 30% depending on the version and horizon. But for the longer maturities the results are different. The proportion stayed between 20% and 86%. In the version with best performance, the external shocks did not explain most of the variance of the forecast, contrary to the other versions.

Table 6 also shoes the proportion of the variance of the survival probability forecasts that is explained with external shocks. In this case the results of all the versions are consistent and point to the importance of the external effect, explaining between 43% to 100% of the variance.

The impulse response on the sovereign rates - 1m, 3y, 20y rates - show that upward shocks of the US short rate gradually increase the Brazilian rates. On

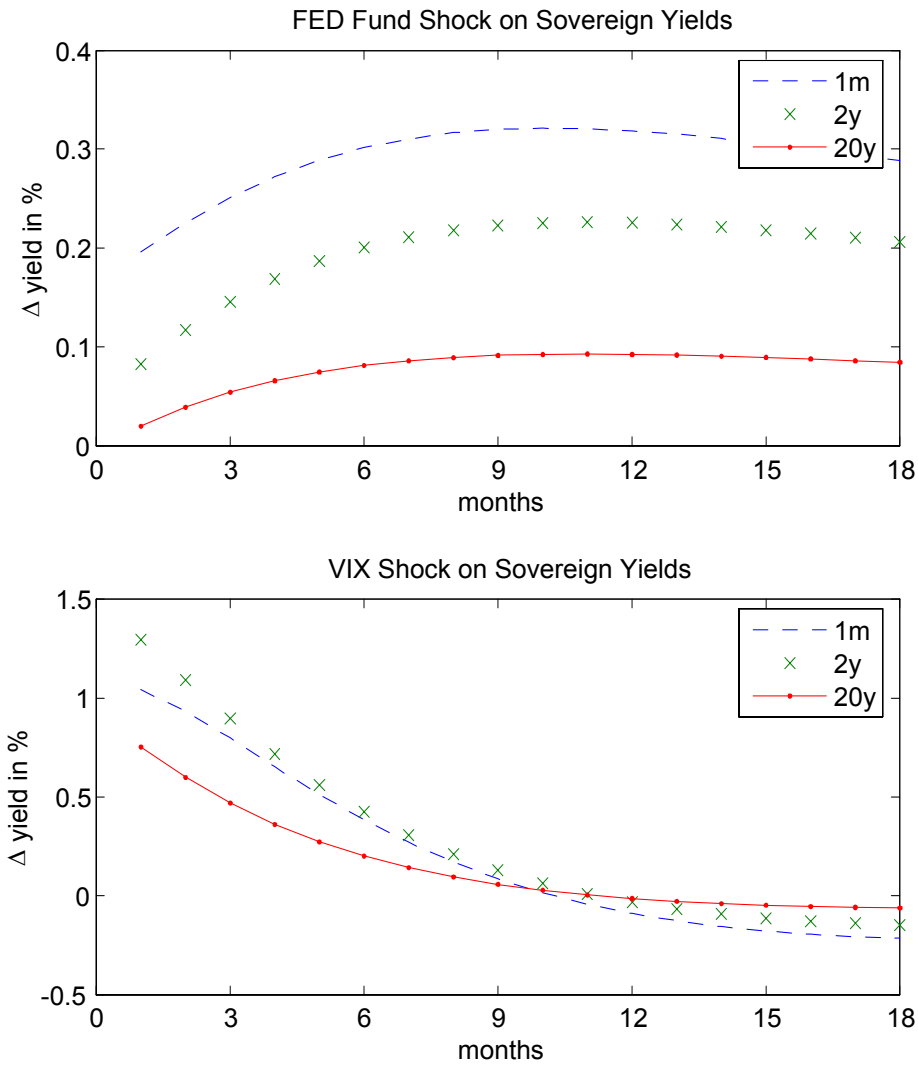

Figure 2: Response of the sovereign yield curve to international macro indicators.

the other hand, a higher VIX have a big but transitory effect on the Brazilian short rate, rising it.

Finally, table 7 evaluates the in-sample fitting of the $\Phi_2^\star$ specification. It shows the measurement errors of the yields of the short, medium and long-term rates.

Tab 7

Measurement errors for selected maturities. Sovereign yields.

	x	$\Phi_1^\star$	Σ_2	$\Phi_2^\star$	$\delta\Phi_2^\star$
1m	0.0165	0.0052	0.0064	0.0179	0.0150
3y	0.0063	0.0105	0.0088	0.0068	0.0084
20y	0.0092	0.0167	0.0056	0.0198	0.0209

6 Conclusion

The interrelation between the interest rates and macroeconomic variables as defined in the macro-finance models proposed by Ang and Piazzesi (2003) evaluates the effect of the long run expectation underlying in the yields of greater maturity on the macro variables and vice versa. However, many articles in the literature present either sub or over-identified models. An exception is Pericoli and Taboga (2006), but here we extended the list of possible identifications, and empirically studied their properties. Although we show that the response of the yield curve to macro shocks is invariant to Dai and Singleton (2000) transformations, the inference can be affected by the choice of the specification. Moreover, we add an affine reduced credit risk model to study the Brazilian sovereign yield curve.

The contributions of this text are: 1) Discuss in a more comprehensive way the identification of the model; 2) analyze two markets; 3) estimate, as an instrument to evaluate the robustness of the results of the macro finance model, a common factor model, which does not have no arbitrage restrictions; 4) we analyzed in the domestic financial market the interrelation between the exchange rate and the expected inflation - measured via BM&F swaps or the *Focus* Central Bank survey - and the yield curve; 5) we analyzed in the sovereign bond market the effect of the US short rate and of VIX volatility on the Brazilian yield curve.

The main results are: 1) the macro finance model presented a similar but inferior performance with respect to the common factor model; 2) in both markets, the incorporated economic variables improved the performance of the model, indicating that the higher complexity was compensated by the higher information; 3) in the case of the domestic market it is shown that great part of the variance of the predicted path can be attributable to the identified nominal shocks, and that the rates of greater maturity are more sensitive to those shocks than the short rate, which is approximately the Selic rate controlled by the Central Bank; 4) in the case of the external market the results are less consistent and not all versions point to the dominance of the external shocks in the sovereign yield curve and the version with the best performance indicated that the greater part

of the variance can be attributed to domestic idiosyncratic factors; 5) however, in all cases the greatest part of the variance of the survival probability forecasts is due to external shocks.

In this text we do not discuss an interpretation of the latent factors, which depend on the adopted identification restriction. An immediate extension would be to relate the identification restrictions to the properties of the latent factors, which would permit the interpretation of all the shocks that affect the path of the yield curve.

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A Appendix: Affine Models with Kalman Filter and FFBS algorithm

A.1

We present the Kalman Filter and the FFBS algorithm of the Dynamic Linear Model (DLM) in which part of the state vector is observed (M), following West and Harrison (1997). We have

$$\begin{array}{l} Y_t = A + BX_t + \sigma e_t,\ e_t \sim N(0, I), \text{ diagonal } \sigma, \\ X_t = \mu + \phi X_{t-1} + \Sigma u_t,\ u_t \sim N(0, I), \\ X_t = [M_t; \theta_t], \end{array} \tag{38}$$

where A and B are given by 16.

The distribution of the parameters is given bellow:

$$\begin{array}{l} \Psi = (\delta_0, \delta_1, \Phi, \mu, \lambda_0, \lambda_1, \Sigma); \\ D_t = \{\Psi, Y_1, ..., Y_t, M_1, ..., M_t\}; \\ \theta_0 \sim N(m_0, C_0) \text{ is given;} \\ \text{Priori of the state variables: } X_t|D_{t-1} \sim N(a_t, R_t); \\ E(X_t|D_{t-1}) = a_t = \mu + \phi m_{t-1}; \\ V(X_t|D_{t-1}) = R_t = \Phi C_{t-1}^{\intercal} \Phi^{\intercal} + V; \\ \text{Forecast of the yields: } (Y_t|D_{t-1}) \sim N(f_t, Q_t); \\ \text{where } f_t = A + Ba_t,\ \ Q_t = BR_tB^{\intercal} + \sigma^{\intercal}\sigma; \\ \text{Posterior of the state variables: } \ (X_t|D_t) \sim N(m_t, C_t); \\ E(X_t|D_t) = m_t = (M_t, m_t^{\theta}); \\ V(X_t|D_t) = C_t = \begin{pmatrix} 0 & 0 \\ 0 & c_t^{\theta} \end{pmatrix}; \\ m_t^{\theta} = a_t^{\theta} + R_t^{\theta} B^{\theta\intercal} Q_t^{-1}(Y_t - f_t); \\ c_t^{\theta} = R_t^{\theta} + R_t^{\theta} B^{\theta\intercal} Q_t^{-1} B^{\theta} R_t^{\theta\intercal}. \end{array} \tag{39}$$

The predictive density of the yields is:

- conditional to X: $L(Y_t|X_t, D_{t-1}) = -\frac{1}{2}[\log|\sigma^{\intercal}\sigma| + (Y_t - A - BX_t)(\sigma^{\intercal}\sigma)^{-1}(Y_t - A - BX_t)]$;
- non conditional: $L(Y_t|D_{t-1}) = -\frac{1}{2}[\log|Q_t| + (Y_t - f_t)Q_t^{-1}(Y_t - f_t)^{\intercal}]$.

A vague prior was used.

A.2

Proof that the pricing equations and impulse response of Y are preserved: we show that $\widetilde{\beta}_n \mapsto L^{\intercal -1}\beta_n$ and $\widetilde{\alpha}_n = \alpha_n$:

$$\widetilde{\beta}_n^{\intercal} = -\widetilde{\delta}_1^{\intercal}(I + \widetilde{\Phi}^{\star} + ... + \widetilde{\Phi}^{\star n}) = -\delta_1^{\intercal} L^{-1}(I + L\Phi^{\star}L^{-1} + ... + L\Phi^{\star n}L^{-1}) = \beta_n^{\intercal} L^{-1} \tag{40}$$

$$\begin{aligned} \widetilde{\alpha}_n &= -\widetilde{\delta}_0 + \widetilde{\alpha}_{n-1} + \widetilde{\beta}_n^{\intercal}(\widetilde{\mu} + \widetilde{\Sigma}\widetilde{\lambda}_0) + \tfrac{1}{2}\widetilde{\beta}_n^{\intercal}\widetilde{\Sigma}\widetilde{\Sigma}^{\intercal}\widetilde{\beta}_n \\ &= -\delta_0 + \widetilde{\alpha}_{n-1} + \beta_n^{\intercal} L^{-1}(L\mu - L\Sigma\lambda_0) + \tfrac{1}{2}\beta_n^{\intercal} L^{-1} L\Sigma\Sigma^{\intercal} L^{\intercal} L^{\intercal -1}\beta_n, \\ &= -\delta_0 + \alpha_{n-1} + \beta_n^{\intercal}(\mu - \Sigma\lambda_0) + \tfrac{1}{2}\beta_n^{\intercal}\Sigma\Sigma^{\intercal}\beta_n = a_n \end{aligned} \tag{41}$$

Since matrices A and B are formed of a_n and $\beta_n^{\intercal}$, respectively, it follows that $\widetilde{A} = A$ and $\widetilde{B} = BL^{-1}$. That is, $Y_t = A + BX_t = \widetilde{A} + \widetilde{B}\widetilde{X_t}$.

The n-period impulse response function of Y is $\beta\Phi^n\Sigma\varepsilon_t$. So, the transformed impulse response is: $\beta L^{-1}(L\Phi L^{-1})^n L\Sigma\varepsilon_t = \beta L^{-1}L\Phi^n L^{-1}L\Sigma\varepsilon_t = \beta\Phi^n\Sigma\varepsilon_t$.

A.3

Proof of the invariance of the likelihood with Kalman Filter under DS transformations. We will show that T_L: $L(Y_t|\Psi) = L(Y_t|\widetilde{\Psi})$ and $L(Y_t|X_t, \Psi) = L(Y_t|\widetilde{X}_t, \widetilde{\Psi})$, where:

$$\begin{aligned} &L = \begin{pmatrix} I & 0 \\ \alpha & \beta \end{pmatrix} \\ &\widetilde{X}_t = LX_t \\ &\widetilde{\Psi} = T_L(\Psi) = (\delta_0, (L^{\intercal})^{-1}\delta_1, L\Phi L^{-1}, L\mu, \lambda_0, \lambda_1 L^{-1}, L\Sigma) \end{aligned} \tag{42}$$

That is, the likelihood using Kalman Filter is preserved under the transformations T_L of the parameters. First, observe that:

$$\begin{aligned} &\widetilde{V} = \widetilde{\Sigma}\widetilde{\Sigma}^{\intercal} = LVL^{\intercal}, \\ &\widetilde{\mu}^{\star} = L\mu^{\star} = L\mu + L\Sigma\lambda_0 = \widetilde{\mu} + \widetilde{\Sigma}\lambda_0, \\ &\widetilde{\Phi}^{\star} = L\Phi^{\star}L^{-1} = L(\Phi + \Sigma\lambda_1)L^{-1} = \widetilde{\Phi} + \widetilde{\Sigma}\widetilde{\lambda}_1. \end{aligned} \tag{43}$$

Next, it is proved by induction that the Kalman Filter equations are preserved by the transformations when it is assumed that $\widetilde{m}_0 = Lm_0$ and $\widetilde{C}_0 = LC_0L^{\intercal}$. We show by induction that for every t,

$$\begin{aligned} \widetilde{m}_t = Lm_t,\ \widetilde{C}_t = LC_tL^{\intercal}, \\ \widetilde{a}_t = La_t,\ \widetilde{R}_t = LR_tL^{\intercal}. \\ \widetilde{f}_t = f_t,\ \widetilde{Q}_t = Q_t \end{aligned} \tag{44}$$

For $t = 1$, we have

$$\begin{aligned} \widetilde{a}_1 = \widetilde{\mu} + \widetilde{\Phi}\widetilde{m}_0 = L\mu + L\Phi L^{-1}Lm_0 = La_1, \\ \widetilde{R}_1 = \widetilde{\Phi}\widetilde{C}_0\widetilde{\Phi}^{\intercal} + \widetilde{V} = L\Phi L^{-1}LC_0L^{\intercal}\,L^{\intercal -1}\Phi L^{\intercal} = LR_1L^{\intercal}, \end{aligned} \tag{45}$$

and,

$$\begin{aligned}\widetilde{f}_1 = \widetilde{A} + \widetilde{B}\widetilde{a}_1 = A + BL^{-1}La_1 = f_1,\\ \widetilde{Q}_1 = \widetilde{B}\widetilde{R}_1\widetilde{B}^\intercal + \widetilde{\sigma} = BL^{-1}LR_1L^\intercal(L^{-1})^\intercal B^\intercal + \sigma = BR_1B^\intercal + \sigma = Q_1.\end{aligned} \tag{46}$$

Then

$$\begin{aligned}\widetilde{m}_1^\theta = \widetilde{a}_1^\theta + \widetilde{R}_1^\theta\widetilde{B}^{\theta\intercal}\widetilde{Q}_1^{-1}(Y_1 - \widetilde{f}_1) = \beta a_1^\theta + \beta R_1^\theta\beta^\intercal\beta^{\intercal-1}B^{\theta\intercal}Q_1^{-1}(Y_1 - f_1) = \beta m_1^\theta,\\ \widetilde{c}_1^\theta = \widetilde{R}_1^\theta + \widetilde{R}_1^\theta\widetilde{B}^\theta\widetilde{Q}_1^{-1}\widetilde{B}^\theta\widetilde{R}_1^{\theta\intercal} = \beta R_1^\theta\beta^\intercal + \beta R_1^\theta\beta^\intercal\beta^{\intercal-1}B^{\theta\intercal}Q_1^{-1}B^\theta\beta^{-1}\beta R_1^{\theta\intercal}\beta^\intercal\\ = \beta c_1^\theta\beta^\intercal.\end{aligned} \tag{47}$$

Hence $\widetilde{m}_1 = Lm_t$ and $\widetilde{C}_t = LC_tL^\intercal$.

Thus, for $t = 1$ the property holds. Now, suppose 44 is true. Then:

$$\begin{aligned}\widetilde{a}_{t+1} = \widetilde{\mu} + \widetilde{\Phi}\widetilde{m}_t = L\mu + L\Phi L^{-1}Lm_t = La_{t+1},\\ \widetilde{R}_{t+1} = \widetilde{\Phi}\widetilde{C}_t\widetilde{\Phi}^\intercal + \widetilde{V} = L\Phi L^{-1}LC_tL^\intercal L^{\intercal^{-1}}\Phi L^\intercal = LR_{t+1}L^\intercal,\end{aligned} \tag{48}$$

$$\begin{aligned}\widetilde{f}_{t+1} = \widetilde{A} + \widetilde{B}\widetilde{a}_{t+1} = A + BL^{-1}La_{t+1} = f_{t+1},\\ \widetilde{Q}_{t+1} = \widetilde{B}\widetilde{R}_{t+1}\widetilde{B}^\intercal + \widetilde{\sigma} = BL^{-1}LR_{t+1}L^\intercal(L^{-1})^\intercal B^\intercal + \sigma = BR_{t+1}B^\intercal + \sigma = Q_{t+1},\end{aligned} \tag{49}$$

$$\begin{aligned}\widetilde{m}_{t+1}^\theta = \widetilde{a}_{t+1}^\theta + \widetilde{R}_{t+1}^\theta\widetilde{B}^{\theta\intercal}\widetilde{Q}_{t+1}^{-1}(Y_{t+1} - \widetilde{f}_{t+1})\\ = \beta a_{t+1}^\theta + \beta R_{t+1}^\theta\beta^\intercal\beta^{\intercal-1}B^{\theta\intercal}Q_{t+1}^{-1}(Y_{t+1} - f_{t+1}) = \beta m_{t+1}^\theta,\\ \widetilde{c}_{t+1}^\theta = \widetilde{R}_{t+1}^\theta + \widetilde{R}_{t+1}^\theta\widetilde{B}^\theta\widetilde{Q}_{t+1}^{-1}\widetilde{B}^\theta\widetilde{R}_{t+1}^{\theta\intercal}\\ = \beta R_{t+1}^\theta\beta^\intercal + \beta R_{t+1}^\theta\beta^\intercal\beta^{\intercal-1}B^{\theta\intercal}Q_{r+1}^{-1}B^\theta\beta^{-1}\beta R_{t+1}^{\theta\intercal}\beta^\intercal = \beta c_{t+1}^\theta\beta^\intercal,\end{aligned} \tag{50}$$

$$\widetilde{m}_{t+1} = Lm_{t+1},\ \widetilde{C}_{t+1} = LC_{t+1}L^\intercal. \tag{51}$$

This proves (44). It follows that $L(Y_t|\Psi) = L(Y_t|\widetilde{\Psi})$ and $L(Y_t|X_t, \Psi) = L(Y_t|\widetilde{X}_t, \widetilde{\Psi})$.

Finally, note that the argument above shows that the likelihood is also invariant to T_O rotations.

A.4

Proof of the invariance of the likelihood with Chen-Scott under DS transformations. The likelihood is

$$\mathcal{L}(\Psi) = \sum_{t=2}^{T}(-\log|\det J| + \log f_X(M_t, \theta_t|M_{t-1}, \theta_{t-1})) \tag{52}$$

$$= \sum_{t=2}^{T}(-\log|\det J| + \log f_X(M_t, \theta_t|M_{t-1}, \theta_{t-1})) \tag{53}$$

$$= (T-1)\log|\det J| - \frac{1}{2}(T-1)\log\det\Sigma\Sigma^\intercal$$

$$-\frac{1}{2}\sum_{t=2}^{T}(X_t-\mu-\Phi X_{t-1})^{\intercal}(\Sigma\Sigma^{\intercal})^{-1}(X_t-\mu-\Phi X_{t-1}) \tag{54}$$

where $J=\begin{pmatrix} I & 0 \\ \beta^M & \beta^\theta \end{pmatrix}$.

Will will show that $\mathcal{L}(\Psi)=\mathcal{L}(T_L\Psi)$, where L is the invariant operator. The third term, when transformed, is unchanged:

$$(LX_t-L\mu-L\Phi L^{-1}LX_{t-1})^{\intercal}(L\Sigma(\Sigma L)^{\intercal})^{-1}(LX_t-L\mu-L\Phi L^{-1}LX_{t-1}) \tag{55}$$

$$=(LX_t-L\mu-L\Phi L^{-1}LX_{t-1})^{\intercal}(L^{-1})^{\intercal}(\Sigma\Sigma^{-1})L^{-1}(LX_t-L\mu-L\Phi L^{-1}LX_{t-1}) \tag{56}$$

$$=(X_t-\mu-\Phi X_{t-1})^{\intercal}(\Sigma\Sigma^{\intercal})^{-1}(X_t-\mu-\Phi X_{t-1}) \tag{57}$$

The first term, when transformed, results in

$$-\frac{1}{2}(T-1)\log\det L\Sigma(L\Sigma)^{\intercal}=-\frac{1}{2}(T-1)[\log\det\Sigma\Sigma^{\intercal}+\log\det L+\log\det L^{\intercal}].$$

$$-\frac{1}{2}(T-1)\log\det\Sigma\Sigma^{\intercal}-(T-1)\log\det L.$$

Now, to calculate the transformed second term $-(T-1)\log|\det J|$, note that

$$\det J=\det\begin{pmatrix} I & 0 \\ B^M & B^\theta \end{pmatrix}=\det B^\theta,$$

and that $(BL^{-1})^\theta=\beta^{-1}B^\theta$ because

$$BL^{-1}=\begin{pmatrix} B^M & B^\theta \end{pmatrix}\begin{pmatrix} I & 0 \\ -\beta^{-1}\alpha & \beta^{-1} \end{pmatrix}.$$

So, the result of applying L will be $-\log|\det\beta^{-1}B^\theta|=-\log|\det B^\theta|-\log|\det\beta^{-1}|=-\log|\det B^\theta|+\log|\det\beta|$. Now, since

$$\det L=\det\begin{pmatrix} I & 0 \\ \alpha & \beta \end{pmatrix}=\det\beta,$$

the $(T-1)\log\det L$ expression of the first two terms of the likelihood will cancel because of the different signs.

A.5

The step 4 of the MCMC requires a realization of $\theta_t^w\sim\theta_t|D_T$, $t=1,..,T$. In what follows, we present a modification of the FFBS algorithm (see West and Harrison) to the case in which part of the latent variables is known.

Sampling of $\theta_t^w \sim \theta_t | D_T, \theta_{t+1}$ is obtained by reverse recursion.

$$\begin{aligned}
&\theta_T^w | D_T \sim N(m_T, C_T). \\
&\theta_t^w \sim N(h_t, H_t), \\
&\text{where } h_t = m_t + B_t(\theta_{t+1} - a_{t+1}) \quad H_t = C_t - B_t R_{t+1} B_t^T \quad B_t = C_t G^T R_{t+1}^{-1}.
\end{aligned} \tag{58}$$

We have for our case in which $X_t = [M_t; \theta_t]$:

$$B_t = \begin{pmatrix} 0 & 0 \\ 0 & c_t^\theta \end{pmatrix} \begin{pmatrix} \phi_{mm} & \phi_{m\theta} \\ \phi_{\theta m} & \phi_{\theta\theta} \end{pmatrix} R_{t+1}^{-1} = \begin{pmatrix} 0 & 0 \\ B_t^{\theta m} & B_t^{\theta\theta} \end{pmatrix}, \tag{59}$$

$$h_t = \begin{pmatrix} M_t \\ m_t^\theta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B_t^{\theta m} & B_t^{\theta\theta} \end{pmatrix} \begin{pmatrix} M_{t+1} - a_{t+1}^m \\ \theta_{t+1}^w - a_t^\theta \end{pmatrix} = \begin{pmatrix} M_t \\ h_t^\theta \end{pmatrix}, \tag{60}$$

$$H_t = \begin{pmatrix} 0 & 0 \\ 0 & c_t^\theta \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ B_t^{\theta m} & B_t^{\theta\theta} \end{pmatrix} \begin{pmatrix} R_t^{mm} & R_t^{m\theta} \\ R_t^{\theta m} & R_t^{\theta\theta} \end{pmatrix} \begin{pmatrix} 0 & B_t^{\theta m \intercal} \\ 0 & B_t^{\theta\theta \intercal} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & H_t^\theta \end{pmatrix}, \tag{61}$$

where

$$h_t^\theta = B_t^{\theta m}(M_{t+1} - a_{t+1}^m) + B_t^{\theta\theta}(\theta_{t+1}^w - a_t^\theta) \tag{62}$$

$$H_t^\theta = B_t^{\theta m} R_t^{mm} B_t^{\theta m \intercal} + 2 B_t^{\theta\theta} R_t^{\theta m} B_t^{\theta m \intercal} + B_t^{\theta\theta} R_t^{\theta\theta} B_t^{\theta\theta \intercal} \tag{63}$$

$$\theta_t^w \sim N(h_t^\theta, H_t^\theta) \text{ repeated for } t = T-1, .., 2. \tag{64}$$

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