# 148

DISCUSSION PAPER

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# ANALYSIS OF EMERGING MARKETS SOVEREIGN CREDIT SPREADS

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Marco S. Matsumura[1]

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# DISCUSSION PAPER

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## SINOPSE

Nosso objetivo é implementar um modelo de apreçamento de títulos soberanos sujeitos a *default* e estimá-lo usando uma série histórica de títulos de mercados emergentes. Usamos um modelo reduzido com uma dinâmica Vasicek 2-fatores sobre dados soberanos brasileiros. A estimação ocorre em duas etapas. Usando Máxima Verossimilhança, primeiro estimamos os parâmetros correspondentes à curva de referência. Em seguida, encontramos as estimativas do conjunto de parâmetros correspondentes à curva do título sujeito a *default* condicionado aos parâmetros da curva livre de *default*. O modelo estimado é usado para calcular a dinâmica da estrutura a termo de taxa de juros, de *spreads* de crédito e de probabilidades de *default*.

## ABSTRACT

Our objective is to implement a credit risk pricing model for sovereign bonds and estimate the model for a historical series of yields of emerging markets bonds. We use a reduced model with a Vasicek 2-factor model on Brazilian sovereign data. The estimation occurs in two stages. Using Maximum Likelihood, we first estimate the parameters corresponding to the reference curve. Then, we find the estimates of the set of parameters corresponding to the defaultable curve conditional on the default-free parameters. The estimated model is used to calculate the dynamics of the term structure of interest rates, of credit spreads and of default probabilities.

# 1 Introduction

The study of Credit Risk begins with the structural models of Black and Scholes (1973) and Merton (1974), which considered bonds issued by firms as options on the firm's assets. In these models, default could only occur at the maturity of the bond. Black and Cox (1976) defined the default as the first time the firm's assets stay below a barrier exogenously specified. Those assets would follow a Brownian Motion. Though many other studies extended this basic setting in various ways, such as using a Levy process to model the evolution of the assets, in all models the price of the defaultable bond will depend on the probability of default and on the expected recovery rate upon default.

An example of a second generation structural model is an interesting work by Leland (1994) and Leland and Toft (1996), which introduces the firm's incentive structure to determine the default barrier endogenously, obtaining as a result the optimal capital structure of the firm. Default occurs when the structure of incentives suggests that it is optimal to the issuer to default or when the payment is impossible. This will happen when the value of the shares fall to zero. A short introductory survey describing the evolution of the credit risk models is given by Giesecke (2004).

However, as we learn from the article by Duffie, Pedersen and Singleton (2003), DPS, the study of sovereign credit risk differs in many aspects from the corporate case. Among the reasons for it, we can cite the following:

- A sovereign debt investor may not have recourse to a bankruptcy code at the default event.

- Sovereign default is mainly a political decision. There exists a trade-off between the costs of making the payments and the costs of reputation, of having the assets abroad seized or of having access to international commerce impeded.

- The same bond can be renegotiated many times. There may not exist cross-default or collective action clauses. Assets of the country can not be used as a collateral.

- There exists a trade off between defaulting on internal or external debt. This may bring interesting implications on the pricing of different classes of assets.
- Also, one must consider the role played by variables such as exchange rates, fiscal dynamics, reserves in strong currency, level of exports and imports, GDP, and many other macro variables.

Thus, constructing a structural model for the case of a country is a more delicate question. It is not obvious how to model the incentive structure of a government and the optimal default decision, or what to consider as being the "assets" of the country which could be seized upon default. Moreover, as we see from the recent Argentinean case, post-default negotiation rounds regarding recovery rate can become very complex and uncertain.

Not surprisingly, then, it is difficult to find structural model papers in the sovereign context. Moreira and Rocha (2004) study the Brazilian case using a 2-factor structural model.

On the other hand, the credit risk literature includes other recent lines of research, including the Reduced Models, where the time of default is a totally inaccessible stopping time with a certain exogenous intensity of default $\lambda$. The definition of a inaccessible stopping time is given in the following. A predictable stopping time $\tau$ is one for which there exists a sequence of announcing stopping times $\tau_1 \leq \tau_2 \leq \ldots$ such that $\tau_n < \tau$ and $\lim \tau_n = \tau$ for all $\omega \in \Omega$ with $\{\tau(\omega) > 0\}$. In the structural models, if the evolution of the assets follows a Brownian diffusion, then the time of default is a predictable stopping time. A stopping time is totally inaccessible if no predictable stopping time $\tau'$ can give any information about $\tau$: $\mathbb{P}[\tau = \tau' < \infty] = 0$. Thus, in the case of the reduced model, the default always comes as a "surprise". This characteristic adds more realism to the modelling, as seen in the case of the MinFins, Russian sovereign bonds, where a price drop of around 80% in the days immediately following the announcement of the default of the Russian domestic bond GKO occurred.

Given the above reasons, we will adopt in this paper a version of the reduced model developed in Duffie and Singleton (1999), DS, as in DPS. The latter analyzes the case of the Russian bonds extending the DS model to include the case of multiple defaults (or multiple "credit events", such as restructuring or renegotiation or change of regime). After estimating the model for the risk free reference curve on a first stage and then for defaultable Russian sovereign bonds on a second, DPS uses the parameters and the model implied spreads to examine, among a number of questions, what are the determinants of the spread, what is the degree of integration between different Russian bonds and what is the correlation between the spreads and the macroeconomic series. That is the paper we followed most closely. Another paper applying reduced model to emerging market is Pagès (2001).

## 2 Duffie-Singleton Reduced Model

The reason for which we will use the DS model is the convenience of using the following formula [see also Lando (1998), who developed the same expression for Cox Processes], which reduces the problem of evaluating the price of an asset subject to default to the case of the pricing of assets without default with an added spread on the spot discount rate. When assets pays upon default a fraction $(1-l)$ of the pre-default market value, DS gives the following expression:

$$P_t = E_t^Q(e^{-\int_t^T r_u du} 1_{[\tau > T]} + W_\tau e^{-\int_t^\tau r_u du} 1_{[\tau \leq T]}) = E_t^Q(e^{-\int_t^T (r_u + s_u) du}), \qquad (1)$$

where $W_\tau = (1 - l_\tau)P_{\tau^-}$ is the recovery rate in case of default, $s(t) = l(t)\lambda(t)$ is the credit spread due to the "expected loss" (the product of the instantaneous probability of default and the loss $l$ upon default), and $E_t^Q$ is the conditional expectation under the equivalent martingale measure $Q$ with respect to the filtration $\mathbb{F}_t$. We consider fixed a probability space $(\Omega, \mathbb{F}, P)$ and assume the existence of a martingale equivalent measure $Q$, that is, arbitrage free markets.

Various assumptions are made in order to make this formula valid. In particular, $\tau$ must be *doubly stochastic* with intensity $\lambda$. This concept is explained briefly in the following. Credit event times are usually modelled as stopping times. So we start with the concept of a point process, which is just a set $\{\tau_i, i \in \mathbb{N}\}$ of stopping times. A counting process $N$ is an associated process that counts the number of events at any given time: $N(t) = \sum_i 1_{[\tau_i \leq t]}$. Now, it can be shown that $N(t)$ is a submartingale. Applying the Doob-Meyer theorem, we know there exists a predictable, nondecreasing process $A(t)$ called the compensator of $N(t)$. One property of the compensator is to give information about the probabilities of the jumps: $E_t[A(t+\Delta t) - A(t)] = E_t[N(t+\Delta t) - N(t)] = P_t[N(t+\Delta t) - N(t) = 1]$. An intensity process $\lambda_t$ for $N(t)$ exists if it is progressively measurable and non-negative such that $A(t) = \int_0^t \lambda(s)ds$. The intuition behind the intensity can be seen in the next expression. Under regularity conditions, it turns out that:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P[\tau \leq t + \Delta t | \tau > t].$$

Finally, the process $N(t)$ with intensity $\lambda$ will be doubly stochastic when $N(t) - N(s)$ conditioned on the entire path of $\lambda$ between $s$ and $t$ follows a Poisson distribution with parameter $\int_s^t \lambda(u)du$. Two valuable textbooks explaining the above concepts in detail are Duffie (2001) and Schönbucher (2003).

At this point, we proceed as following. We make a choice of an Ito diffusion for $(r_t, s_t)$ and use Feynman-Kac to obtain an expression for the price of the bond as a function of the parameters of the diffusion. Subsequently, we find the likelihood function of the historical series of yields and then estimate the parameters by maximum likelihood. It is more convenient, however, to divide this procedure in two steps, first making the estimation of the parameters relative to $r_t$ using the reference (default free) curve, for which the convention is to use US Treasury bonds or swaps from the LIBOR markets, and then the estimation of the rest of the parameters using the emerging countries bonds.

## 3 Reference Curve

In this section we explain how the parameters of the reference curve will be estimated.

The price of a security without credit risk that pays one unity at the maturity date $T$ is $P_t = E_t^Q(e^{-\int_t^T r_u du})$.

We are going to use the Vasicek model, according to which the instantaneous interest rates is given by

$$dr_t = K^{rr}(\theta^r - r_t)dt + \Sigma^{rr} dw_t^r \tag{2}$$

under the $P$ measure, or

$$dr_t = K^{rr}(\theta^r - r_t)dt + \Sigma^{rr}(dw_t^{r*} - \lambda^r dt) \tag{3}$$

under the $Q$ measure.

In the Vasicek model, the yield of the bond has a closed formula. By Feynman-Kac, if $v$ is a function such that $E_t^Q(e^{-\int_t^T r_u du}) = v(r_t, t, T)$, then $v(r, t)$ must satisfy the following PDE:

$$\frac{\partial v}{\partial t}(r,t) + \frac{1}{2}\Sigma^2 \frac{\partial^2 v}{\partial r^2}(r,t) + (K^{rr}\theta^r - \lambda^r \Sigma^{rr} - K^{rr} r)\frac{\partial v}{\partial r}(r,t) - rv(r,t) = 0, \tag{4}$$

with the initial condition $v(r, T) = 1$.

It turns out that a solution can be obtained supposing that it is of the form $v(r_t, t, T) = e^{m(t,T) - n(t,T) r_t}$ and substituting it into the PDE. Separating the part with $r$ and the independent part of the resulting equation, the following system of equations arises:

$$n_t(t,T) = K^{rr} n(t,T) - 1, \tag{5}$$

$$m_t(t,T) = (K^{rr}\theta^r - \lambda^r \Sigma^{rr}) n(t,T) - \frac{1}{2}\Sigma^{rr2} n^2(t,T), \tag{6}$$

with initial condition $n(T, T) = 0$ e $m(T, T) = 0$. See Duffie and Kan (1996) or Duffie (2001) for details.

Solving that system of ODE's, one explicitly obtains the value of the bond as a function of the state variable $r$ and of the parameters to be estimated using market prices, or equivalently, yields, which contain the same amount of information. The vector $c = (c_1, \ldots, c_n)$ will denote the historical series of yields. It must be remembered that the yields will be the only observable variables of the problem. The state variables $r$ and $s$ are not directly observable (although the 1-month T-Bill of the American Treasure could be a reasonable approximation to $r$ and could be used to test the results of the estimation).

The zero-coupon yield is then given by

$$c_t = y(t,T) = \frac{n(t,T)r_t - m(t,T)}{T-t} = g(r_t,t,T). \quad (7)$$

In practice, bonds used for the estimation have coupons and may have other features like amortization and collateral, as in the case of C-Bonds. We choose to make an approximation of considering the coupon bond as a "fictitious" zero-coupon bond with yield and maturity given by the yield-to-maturity and duration of the coupon bond. There are obvious problems associated with this approximation, specially when one realizes that the duration of the bond varies with the time. It may even happen, in a sufficiently long series, that the duration of the bond in the beginning of the series lies before the end of the series. There are, however, many possible improvements that can be made, and the results of the present estimation will be the basis with which ongoing and future work will be compared. It is interesting to know whether the approximation made would suffice, given the amount structure that needs to be imposed in the complete model and the available set of data available for the estimation process. Also, for the planned extensions to other directions a simpler model would be a good starting point.

As previously mentioned, to estimate the parameters of the model, maximum likelihood estimators is used. The likelihood is given by the density function of the vector $c$ of yields and of the vector $\psi = (K^{rr}, \theta^r, \lambda^r, \Sigma^{rr})$ of the parameters, $f_c(c_1, \ldots, c_n; \psi)$. Using Change of Variables formula from $c_t$ to $r_t$, we have:

$$f_c(c_1, \ldots, c_n; \psi) = f_r(h(c_1), \ldots, h(c_n); \psi)|\mathrm{Det}\nabla h| = \prod_{i=1}^{n} f_{r_i|r_{i-1}}(h(c_i); \psi)\frac{T-t_i}{n(t_i)},$$

where $h = g^{-1}$. Note the use of the Markovian property of the diffusion to decompose the density.

The parameters are then estimated by the maximization of the log-likelihood $L$, that is, $\hat{\psi} = \max_\psi L(c,\psi) = \max_\psi \log f_c(c_1, \ldots, c_n; \psi)$. To this end, however, it is necessary to find the transition density function of $r_i$ given $r_{i-1}$. In the case of the specification given to $r$ in equation (2), we can integrate $dr_t$ and conclude that

$r_i | r_{i-1} \sim N(\mu_i, \sigma_i^2)$, where

$$\mu_i = \theta^r - \frac{\lambda^r \Sigma^{rr}}{K^{rr}} + (r_{i-1} - (\theta^r - \frac{\lambda^r \Sigma^{rr}}{K^{rr}}))e^{K^{rr}(t_i - t_{i-1})},$$
$$\sigma_i^2 = \frac{\Sigma^{rr2}}{2K^{rr}}(1 - e^{-2K^{rr}(t_i - t_{i-1})}).$$

Thus, we are able to explicitly find the log-likelihood of the vector $c$ so that maximization by numerical routines is possible. See for instance Pearson and Sun (1994), which implements the Cox-Ingersoll-Ross model. The maximization was first done on Mathematica and then on Matlab. It may be useful to remark that our experience indicated that the first one is a great platform to begin the development, and the graphics of the paper were produced by that program. However, Matlab is much faster in numerical aspects. In any case, using both programs made possible to check one implementation with respect to the other.

There is another point we would like to stress. In order to obtain the likelihood, one must calculate the transition density under the $P$-measure, using the Brownian motion under the objective measure, in contrast to the pricing equation, which has to be evaluated under the $Q$-measure, or else the result of the estimation would be as if the investors were all risk-neutral with respect to the interest rate risk.

The Maximum Likelihood Method produces asymptotically consistent, non-biased and normally distributed estimators:

$$\text{When } T \to \infty, \hat{\psi} \to \psi \text{ a.s.}, \tag{8}$$

$$T^{\frac{1}{2}}(\hat{\psi} - \psi) \to N(0, \Omega) \text{ in distribution}, \tag{9}$$

where

$$\Omega^{-1} = E\left(\frac{\partial L(c;\psi)}{\partial \psi} \frac{\partial L(c;\psi)}{\partial \psi}^T\right) = -E\left(\frac{\partial^2 L(c;\psi)}{\partial \psi^2}\right) \tag{10}$$

using the information inequality. Confidence intervals for the estimators can be constructed using those properties. Note that to calculate $\Omega$ we would have to know the true parameters. An estimator for $\Omega^{-1}$ is the empirical hessian

$$\hat{\Omega}^{-1} := -\frac{1}{n}\sum_{t=1}^{n}\left(\frac{\partial^2 L_t(c;\hat{\psi})}{\partial \psi^2}\right),$$

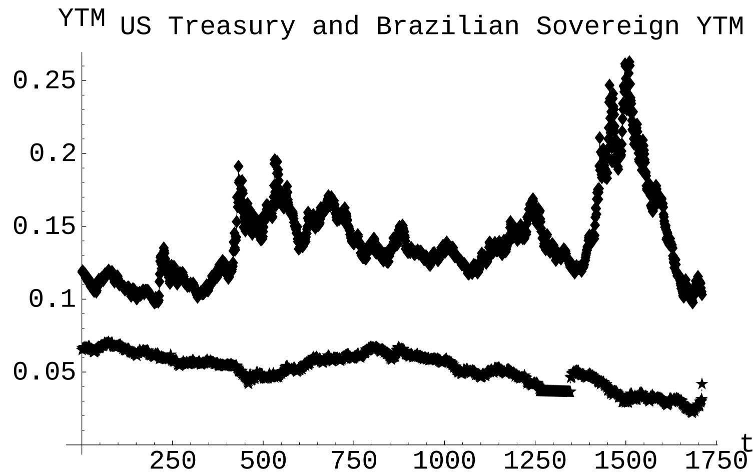



Figure 1: Data used for estimation.

where $L_t$ represents the likelihood of the vector with $t$ elements. More details can be found in Davidson and Mackinnon (1993, Chapter 8).

## 4 Results of the estimation of the reference curve

The data used to construct the reference curve comprises of a series of daily observations from 1997 to 2003 obtained from a Bloomberg terminal (Figure 1), the sole source of information we used to make all the estimations. The results of the estimation together with the confidence intervals are:

| | |
|---|---|
| $K^{rr}$ | $0.3070 \pm 0.0383$ |
| $\theta^r$ | $0.0227 \pm 0.0460$ |
| $\lambda^r$ | $-0.5341 \pm 1.4262$ |
| $\Sigma^{rr}$ | $0.0231 \pm 0.0019$ |

If the empirical hessian used to compute the confidence intervals is normalized, one obtains the correlation matrix of the parameters:

| | $K^{rr}$ | $\theta^r$ | $\lambda^r$ | $\Sigma^{rr}$ |
|---|---|---|---|---|
| $K^{rr}$ | 1. | $-0.027$ | $-0.048$ | 0.979 |
| $\theta^r$ | $-0.027$ | 1. | 0.922 | $-0.027$ |
| $\lambda^r$ | $-0.048$ | 0.922 | 1. | $-0.046$ |
| $\Sigma^{rr}$ | 0.979 | $-0.027$ | $-0.046$ | 1. |

The high correlation among the parameters is surely associated with high numerical instability, which could be caused by a number of factors: 1) the need of a longer historical series, 2) the fictitious zero-coupon bond approximation, or 3) the choice of the Vasicek model, which has a constant volatility. One way to alleviate the problems would be to fix $\theta^r$, the long term instantaneous interest rate, using the average value of 1-month Treasury Bonds yields, thus reducing the dimension of the set of parameters, which would make correlation diminish. Also, the confidence intervals reveal weak results for $\theta^r$ and $\lambda^r$.

Moreover, further experiments aiming those weaknesses of the estimation would include the following:

- Usage of better approximations. The literature suggests many, such as the Nelson and Siegel (1987) approach. Also, we could use a pricing formula with variable maturity date, so that the duration series is used together with the yield to maturity series.

- Usage of full coupon-bond pricing and dynamics.

- More types of data, such as Global Bonds, or longer series.

- Construction of a proper zero-coupon Yield curve from market data and then using the historical series of one fixed maturity to estimate the model. In fact, after the estimations presented here were completed, the author became aware of a rich source of data publicly provided by the Federal Reserve.

- Specifications for the instantaneous interest rates other than Vasicek, such as CIR and including stochastic volatility.

Once the parameters are found we can plot the implied instantaneous interest rates (Figure 2). As previously noted, the parameters of the instantaneous rates were estimated by the maximization of the density of the joint distribution of the evolution of the yields under the objective measure $P$. We also plot what would be the instantaneous interest rate under $Q$ (Figure 3) to illustrate one point of the martingale methodology. The value of an asset is not the discounted flow of dividends. Agents are risk-averse. The value of an asset is the discounted flow of dividends in a constructed "$Q$-world" in which agents would be risk neutral and probabilities are changed to "worsened" conditions, thus compensating for the neutrality.

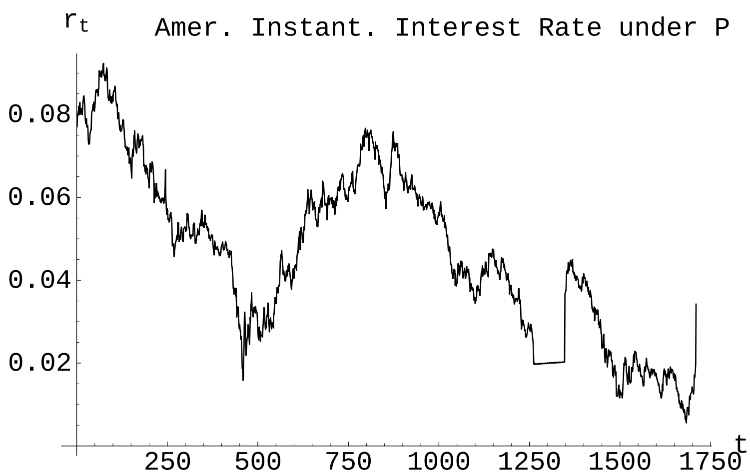



Figure 2: Graphics of $r_t$ in the "real-world" measure.

Graphics of the likelihood around the maximum varying one parameter at a time are given to illustrate qualitatively the maximization (Figures 4-7).

The highly correlated parameters constitute a problem that the model and data could not solve. So 3D and contour plots (Figures 8 and 9) may help illustrate this point.

After that, we show examples of model and real American Term Structure (Figures 10 and 11).

Also, using data available on the Federal Reserve Web address, we plotted the

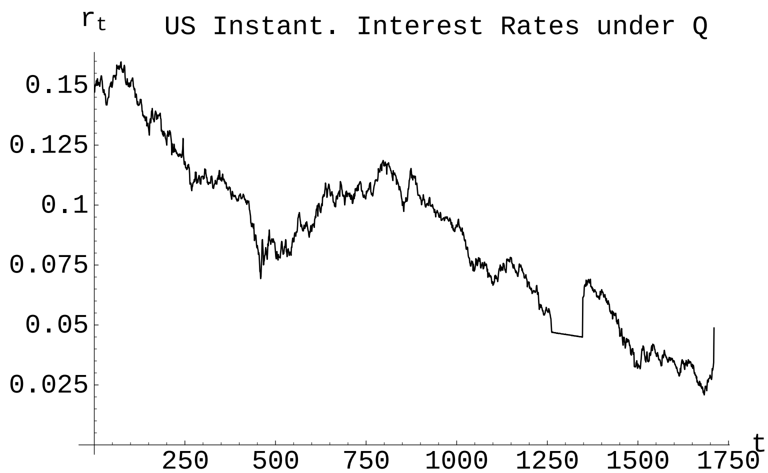



Figure 3: Graphics of $r_t$ in the "Q-world".

evolution of the entire term structure from 1997 to 2003. This is to compared to the 3D graph of the model implied term structure that follows (Figure 12).

We conclude the Reference Curve estimations showing graphics of series of yields of 1-month and 3-month Treasuries from the Fed source (Figures 13 and 14), and of the direct comparison between the model instantaneous rate and the 3-month series (Figure 15). The Fed 1-month series could not be used because it only begins on 2001.

## 5 Spread

Now we can begin to determine the term structure of the credit spreads of the sovereign bonds. A bi-dimensional gaussian process (Vasicek) was chosen for the joint evolution of $(r_t, s_t)$,

$$\begin{bmatrix} dr_t \\ ds_t \end{bmatrix} = \begin{bmatrix} K^{rr} & 0 \\ K^{sr} & K^{ss} \end{bmatrix} \left( \begin{bmatrix} \theta^r \\ \theta^s \end{bmatrix} - \begin{bmatrix} r_t \\ s_t \end{bmatrix} \right) dt + \begin{bmatrix} \Sigma^{rr} & 0 \\ \Sigma^{sr} & \Sigma^{ss} \end{bmatrix} dw_t^{r,s} \qquad (11)$$

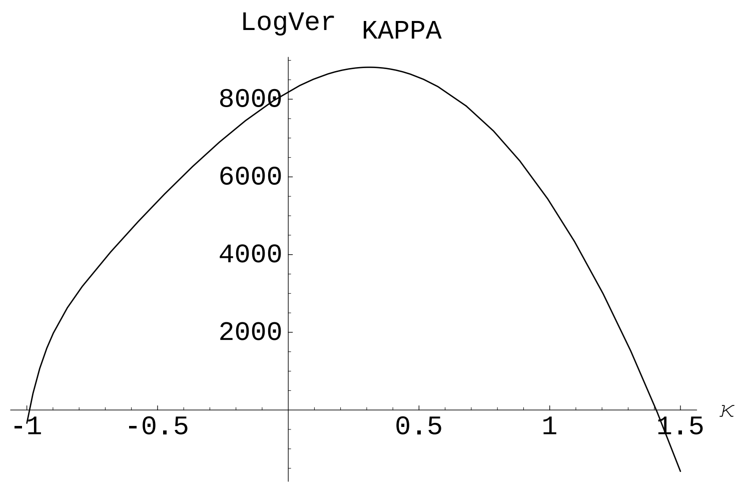



Figure 4: Variation of the log-likelihood changing $\kappa^{rr}$ around the maximum.

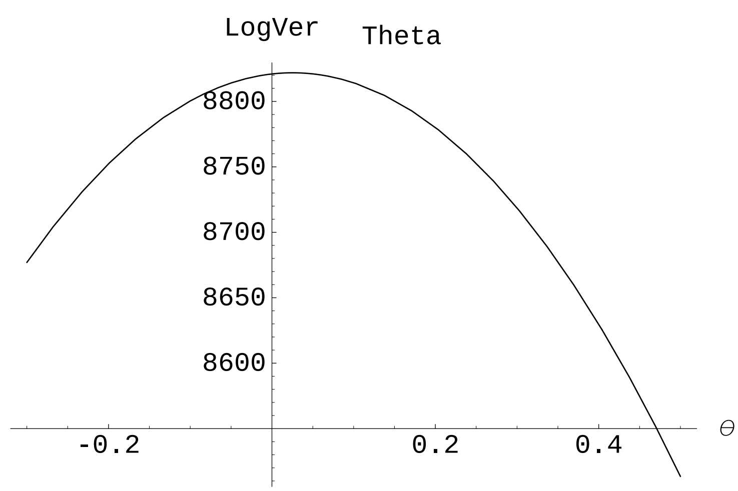



Figure 5: Variation of the log-likelihood changing $\theta^{r}$ around the maximum.

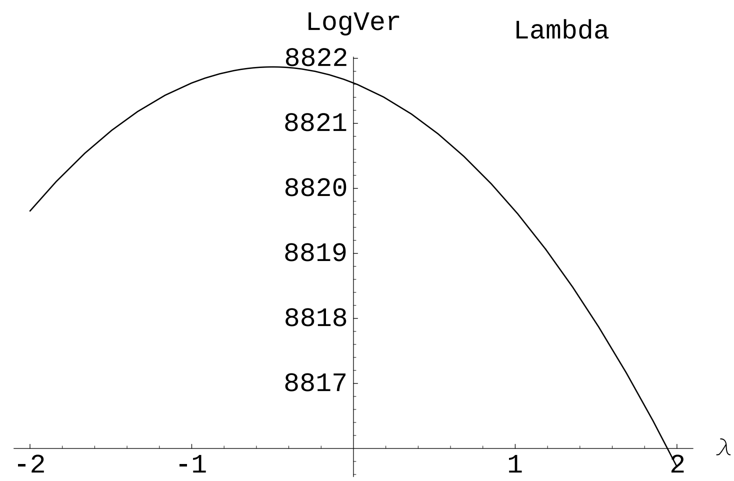



Figure 6: Variation of the log-likelihood changing $\lambda^r$ around the maximum.

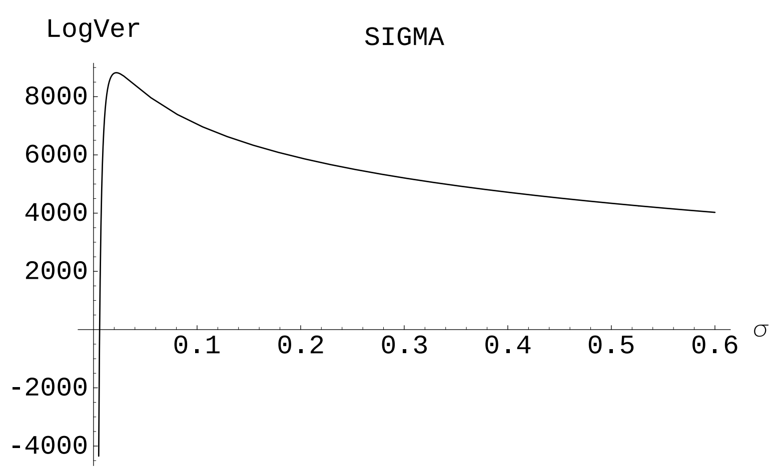



Figure 7: Variation of the log-likelihood changing $\Sigma^{rr}$ around the maximum.

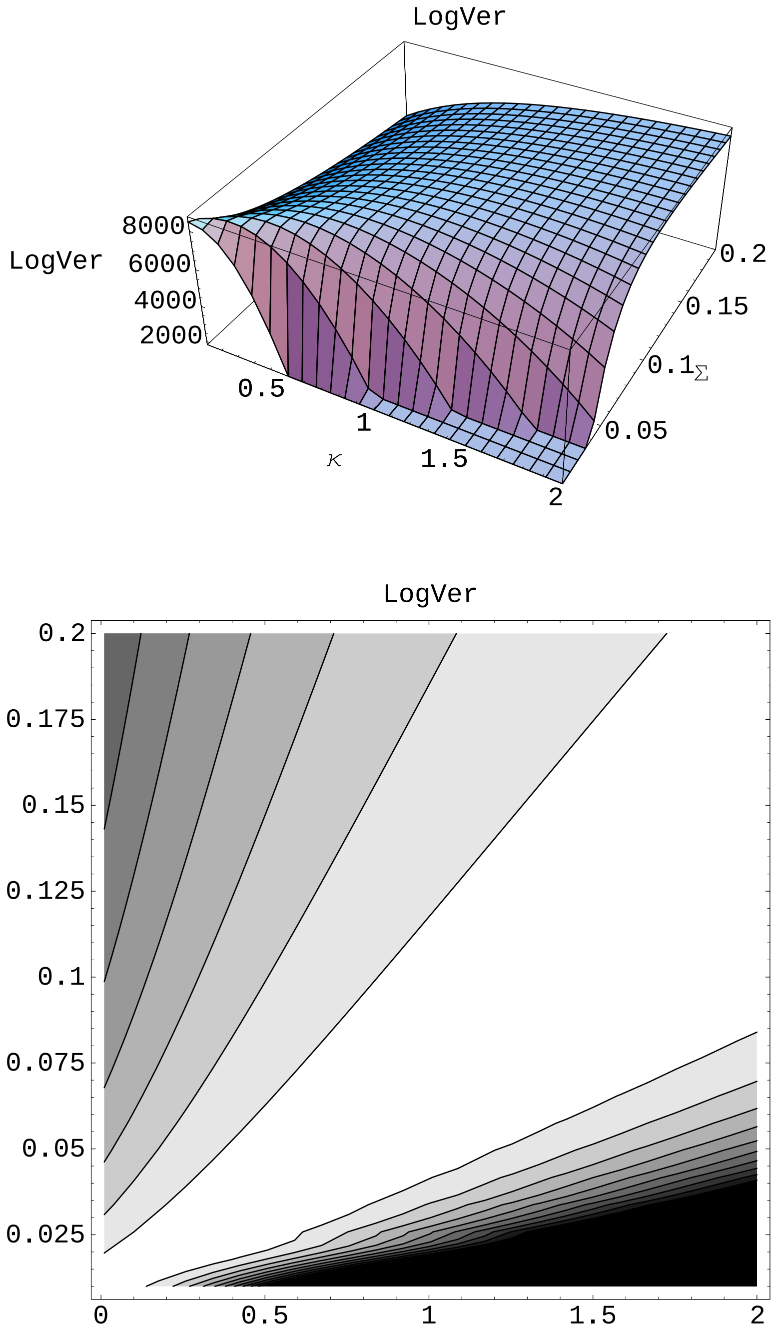



Figure 8: 3D and contour graph of the log-likelihood when varying $K^{rr} \times \Sigma^{rr}$ around the maximum.

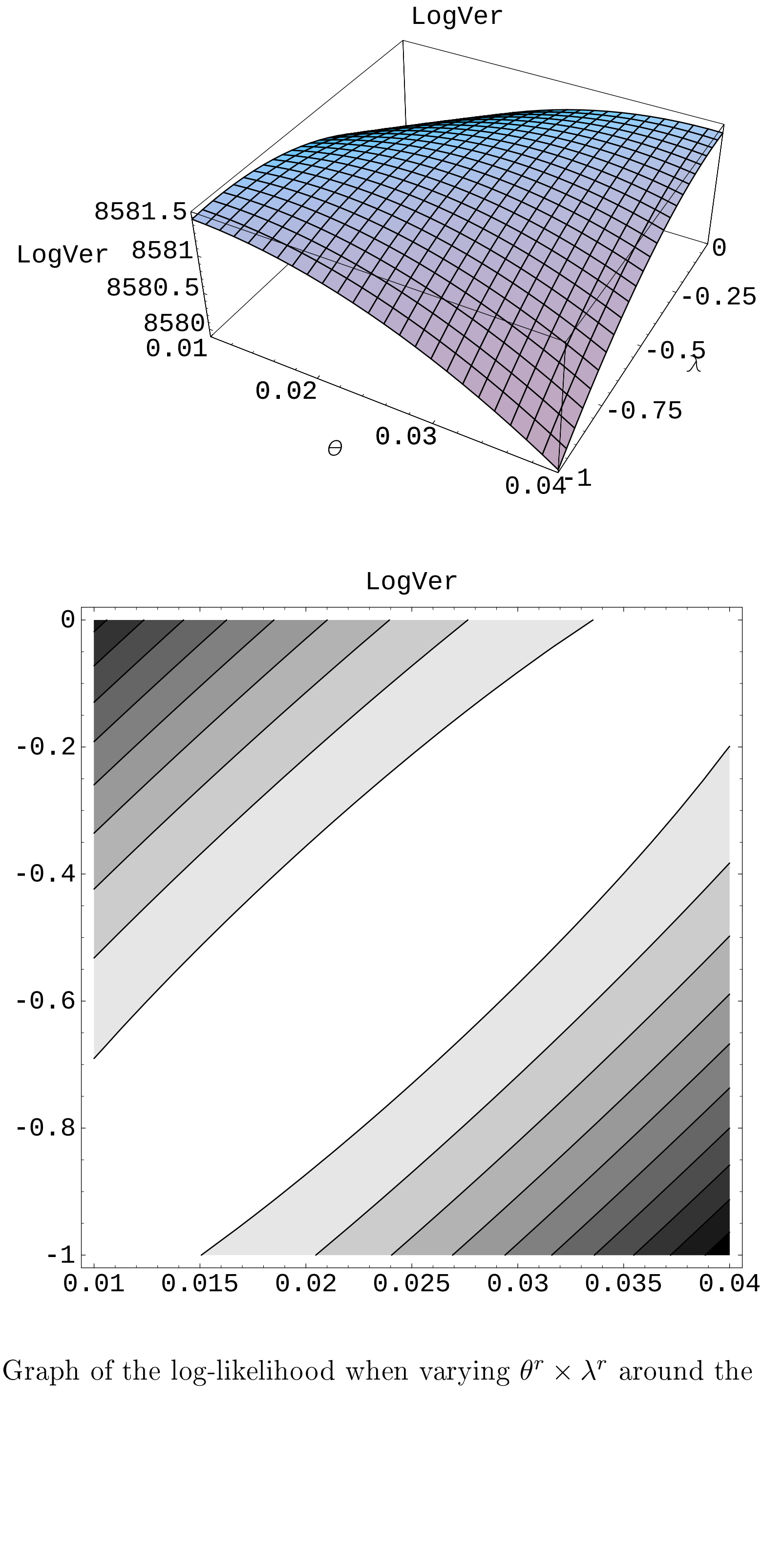



Figure 9: Graph of the log-likelihood when varying $\theta^r \times \lambda^r$ around the maximum.

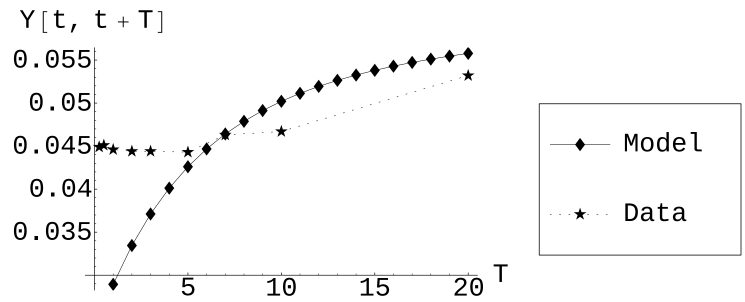



Figure 10: US Term Structure in December $1^{st}$ 1998 — comparison with Fed data.

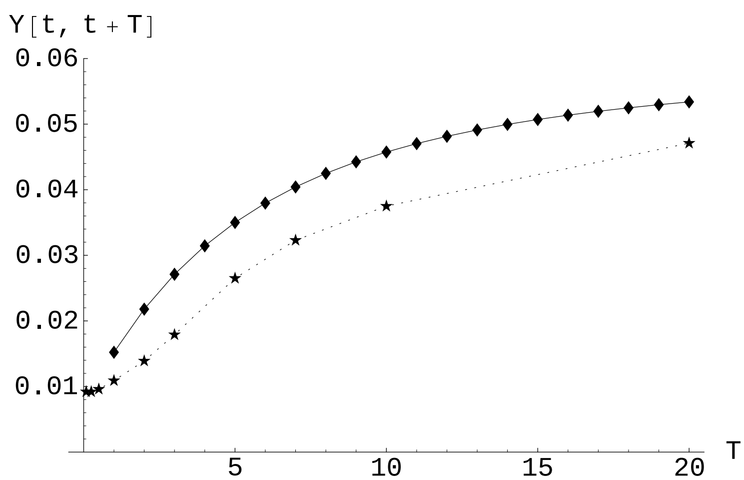
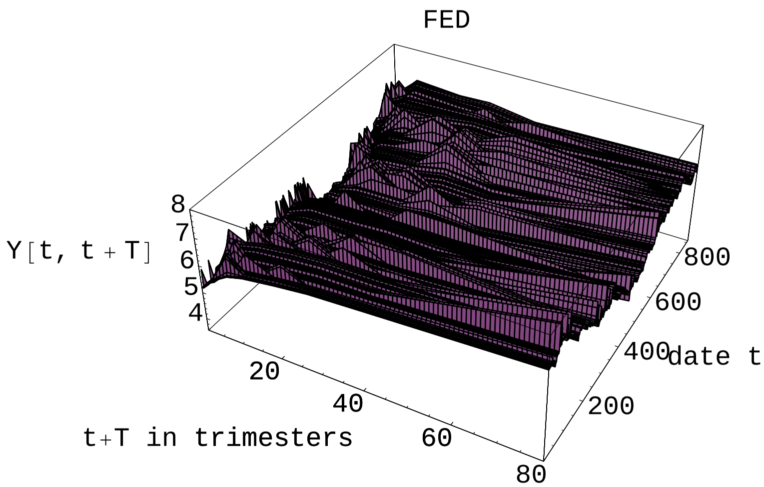



Figure 11: US Term Structure in July $8^{th}$, 2003 — comparison with Fed data.



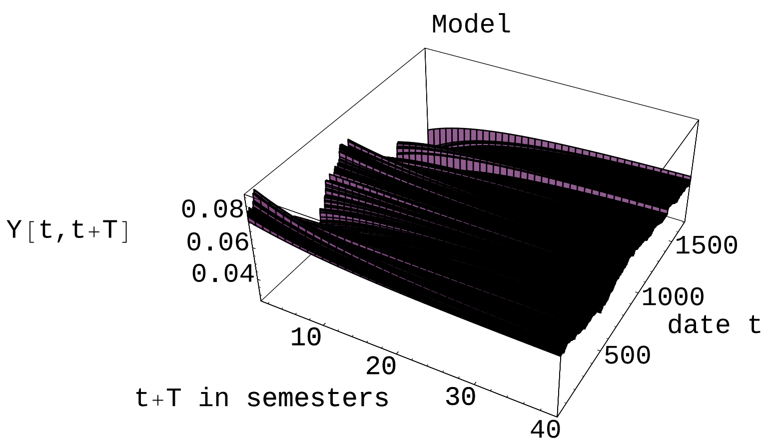




Figure 12: US Term Structure from 1997 to 2003 - Fed × Model.

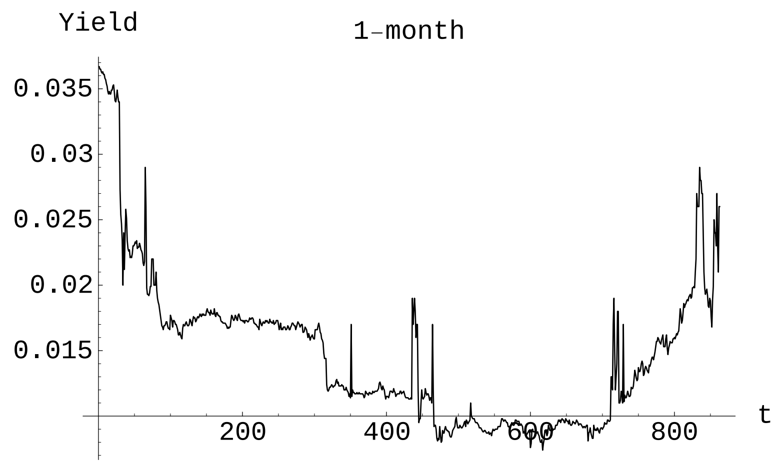



Figure 13: Yields of 1-month Treasury 2001-2005 using data given by the Fed. The series goes back only to 2001.

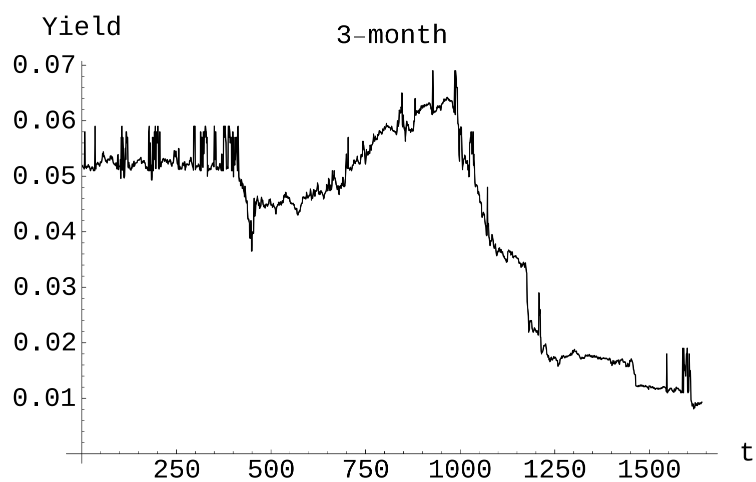



Figure 14: Yields of 3-month Treasury corresponding to the period of the C-Bond data used (1997-2003). Data given by the Fed.

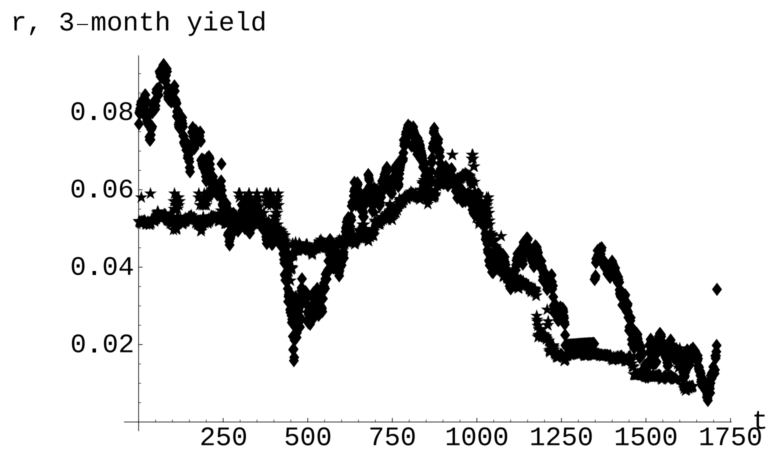



Figure 15: Yields of the 3-month Treasury compared with the instantaneous rate.

where $dw_t^{r,s}$ is a standard 2-Dimensional Brownian Motion under $P$. Change to Brownian Motion under the measure $Q$ is done using Girsanov:

$$dw_t^{r,s} = -\begin{bmatrix}\lambda^r \\ \lambda^s\end{bmatrix} dt + d\tilde{w}_t^{r,s},$$

where $\tilde{w}_t^{r,s}$ is a standard Q-Brownian Motion and $\lambda = (\lambda^r, \lambda^s)$ are market prices of risk.

The choice of the Gaussian process was due to purely pragmatic reasons, for it permits closed formulas for the transition densities that are present in the likelihood, and may require less observations to the convergence of the estimator. Also, it is obviously easier to implement. Models with more parameters and state variables may produce better fitting but at the risk of demanding historical series too long to be obtainable in Emerging Markets. In any case, we are considering to extend the present work to include a third factor, in line with the latest results in the literature [see Dai and Singleton (2000)], which suggests that the ideal number of state variables is 3. Incidentally, on the empirical side, Litterman and Scheinkman (1991) found, using principal component analysis, that 3 factors ("level","slope", and "curvature") account for around 97% of price variations. Finally, we quote Dai and

Singleton (2000): "Researchers are inevitably confronted with trade-offs between the richness of econometric representations of the state variables and the computational burdens of pricing and estimation."

We can now use multidimensional Feyman-Kac into the Duffie-Singleton formula. If $E_t^Q(e^{-\int_t^T (r_u+s_u)du}) = v(r_t, s_t, t, T)$, then $v(r, s, t)$ must satisfy the following PDE:

$$\mathbb{D}v(r,s,t) - (r+s)v(r,s,t) = 0, \tag{12}$$

$$v(r,s,T) = 1.$$

The operator $\mathbb{D}$ is given by

$$\mathbb{D}v(r,s,t) := v_t(r,s,t) + v_{r,s}(r,s,t) \cdot \mu(r,s,t) + \frac{1}{2}\mathrm{tr}[\Sigma\Sigma^T v_{r,r,s,s}],$$

where

$$\mu(r,s,t) = \begin{bmatrix} K^{rr} & 0 \\ K^{sr} & K^{ss} \end{bmatrix} \left( \begin{bmatrix} \theta^r \\ \theta^s \end{bmatrix} - \begin{bmatrix} r_t \\ s_t \end{bmatrix} \right) - \begin{bmatrix} \Sigma^{rr} & 0 \\ \Sigma^{sr} & \Sigma^{ss} \end{bmatrix} \begin{bmatrix} \lambda^r \\ \lambda^s \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} \Sigma^{rr} & 0 \\ \Sigma^{sr} & \Sigma^{ss} \end{bmatrix}.$$

Again, we search for a exponential affine in the state variables solution:

$$v(r,s,t,T) = e^{\alpha(t,T)+\beta(t,T)\cdot(r,s)}. \tag{13}$$

Substituting (13) into the PDE (12) and observing that the resulting expression must be equal to zero for all $r$ and $s$, one obtains

$$\alpha_t = -\beta \cdot K_0 - \frac{1}{2}[(\Sigma^{rr}\beta_1 + \Sigma^{sr}\beta_2)^2 + \beta_2^2 \Sigma^{ss2}], \tag{14}$$

and

$$\beta_t = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} K^{rr} & K^{sr} \\ 0 & K^{ss} \end{pmatrix} \beta. \tag{15}$$

with boundary conditions $\alpha(T,T) = 0$ and $\beta(T,T) = 0$.

Denote by $b = (b_1, \dots, b_n)$ the vector of the yields of the sovereign emerging markets bonds subject to default, which are the observed variables. In terms of the state variables, the yield can be written as

$$y(t,T) = b_t = -\frac{\alpha(t,T) + \beta_1(t,T)r_t + \beta_2(t,T)s_t}{T-t}. \tag{16}$$

Inverting with respect to $s$, we have

$$s_t = \frac{-(T-t)b_t - \alpha(t,T) - \beta_1(t,T)r_t}{\beta_2(t,T)} = H(b_t, r_t). \tag{17}$$

As a result, we can write the density function of $b$ in terms of the density of $s$, which we know how to calculate. Denoting by $f_b$ and $f_s$ the densities of $b$ and $s$, and by $\phi$ the parameters relative to the spread (that is, all the parameters excluding $\psi$), we have

$$\begin{aligned} f_b(b;\psi,\phi,c) &= f_s(H(b_1,r_1),\dots,H(b_n,r_n);\psi,\phi,r)\left|\frac{T-t_1}{\beta_2(t_1,T)}\cdots\frac{T-t_n}{\beta_2(t_n,T)}\right| \\ &= \prod_{i=2}^{n}\left[f_{s_i|(r_{i-1},s_{i-1})}(H(b_i,r_i);\psi,\phi,r)\left|\frac{T-t_i}{\beta_2(t_i,T)}\right|\right]. \end{aligned} \tag{18}$$

As in the previous case, to be able to maximize the log-likelihood given by $L(b;c,\psi,\phi) = \log f_b(b;c,\psi,\phi)$ we need to find the conditional density $f_{s_i|(r_{i-1},s_{i-1})}$. This is done integrating the equation (11). The result is

$$\begin{pmatrix} r_t \\ s_t \end{pmatrix} = E + A + B,$$

where

$$\begin{aligned} E &= P\begin{pmatrix} e^{-K^{rr}t} & 0 \\ 0 & e^{-K^{ss}t} \end{pmatrix} P^{-1}\begin{pmatrix} r_0 \\ s_0 \end{pmatrix}, \\ A &= P\begin{pmatrix} 1-e^{K^{rr}t} & 0 \\ 0 & 1-e^{-K^{ss}t} \end{pmatrix} P^{-1}\left[\begin{pmatrix} \theta^r \\ \theta^s \end{pmatrix} - \begin{pmatrix} \Sigma^{rr} & 0 \\ \Sigma^{sr} & \Sigma^{ss} \end{pmatrix}\begin{pmatrix} \lambda^r \\ \lambda^s \end{pmatrix}\right], \\ B &= P\begin{pmatrix} e^{-K^{rr}t} & 0 \\ 0 & e^{-K^{ss}t} \end{pmatrix}\int_0^t \begin{pmatrix} e^{K^{rr}u} & 0 \\ 0 & e^{K^{ss}u} \end{pmatrix} P^{-1}\begin{pmatrix} \Sigma^{rr} & 0 \\ \Sigma^{sr} & \Sigma^{ss} \end{pmatrix}\begin{pmatrix} d\tilde{w}_u^r \\ d\tilde{w}_u^s \end{pmatrix}. \end{aligned}$$

Thus, the conditional distribution of $s$ is gaussian with know mean and variance and we can proceed to the estimation of the parameters $\phi$.

# 6 Results of the estimation of the term structure of the spreads

The first results of the maximization of the sovereign yield curve produced highly correlated parameters. Once more, this could be an indication that the data used is insufficient for the estimation. However, some adaptations greatly improved the results. First, we set $\Sigma^{sr} = 0$ after noticing that it did not alter the value of the maximized likelihood. We show below the results at this point:

| | |
|---|---|
| $K^{sr}$ | $0.5811 \pm 0.0629$ |
| $K^{ss}$ | $0.2726 \pm 0.0344$ |
| $\theta^s$ | $0.0269 \pm 0.2115$ |
| $\lambda^s$ | $-1.1003 \pm 1.5194$ |
| $\Sigma^{sr}$ | 0 |
| $\Sigma^{ss}$ | $0.1034 \pm 0.0082$ |

The corresponding correlation matrix is:

| | $K^{sr}$ | $K^{ss}$ | $\theta^s$ | $\lambda^s$ | $\Sigma^{ss}$ |
|---|---|---|---|---|---|
| $K^{sr}$ | 1. | 0.6117 | -0.0320 | 0.0212 | 0.5975 |
| $K^{ss}$ | 0.6117 | 1. | -0.0446 | 0.0606 | 0.9767 |
| $\theta^s$ | -0.0320 | -0.0446 | 1. | 0.8762 | -0.0441 |
| $\lambda^s$ | 0.0212 | 0.0606 | 0.8762 | 1. | 0.0608 |
| $\Sigma^{ss}$ | 0.5975 | 0.9767 | -0.0441 | 0.0608 | 1. |

Clearly, the correlation matrix continues to exhibit high values. For instance, the high correlation between $\theta^s$ and $\lambda^s$ means that the model and data could not separate the long term instantaneous spread $\theta^s$ from the associated risk premium $\lambda^s$. So, it was necessary to reduce for a second time the dimension of the problem.

We fixed $\lambda^s$ so that $\lambda^s \frac{\sigma^{ss}}{\kappa^{ss}} = \lambda^r \frac{\sigma^{rr}}{\kappa^{rr}}$. The idea is that the risk premium comes from the risk aversion of the agents from the market, so that if an agent demands a certain level of premium to buy US bonds, the agent would ask at least the same premium for emergent market bonds, considering its greater riskiness. Thus, the true $\lambda^s$ is uncertain, so we normalize the $\lambda$'s to equalize the level of riskiness per amount of volatility.

The final result of the maximization of the likelihood with respect to $\phi$ is given in the table below, together with the confidence intervals, always taken from the empirical hessian of the likelihood:

| | |
|---|---|
| $K^{sr}$ | $0.5810 \pm 0.0629$ |
| $K^{ss}$ | $0.2730 \pm 0.0344$ |
| $\theta^s$ | $0.1367 \pm 0.0996$ |
| $\lambda^s$ | $-0.1074$ |
| $\Sigma^{sr}$ | $0$ |
| $\Sigma^{ss}$ | $0.1035 \pm 0.0083$ |

The corresponding correlation matrix of the estimators is:

| | $K^{rs}$ | $K^{ss}$ | $\theta^s$ | $\Sigma^{ss}$ |
|---|---|---|---|---|
| $K^{rs}$ | 1. | 0.0385 | 0.0015 | 0.0376 |
| $K^{ss}$ | 0.0385 | 1. | 0.0002 | 0.0336 |
| $\theta^s$ | 0.0015 | 0.0002 | 1. | 0.0006 |
| $\Sigma^{ss}$ | 0.0376 | 0.0336 | 0.0006 | 1. |

We plot the evolution of the instantaneous spreads given by the model with those estimated parameters (Figure 16) . Then, we show the model implied term structure on January $10^{th}$, 2003. Using data from J.P. Morgan, an approximated Brazilian Sovereign term structure for the same day was constructed, displaying spreads over treasury for given durations, to compare with the results of the model (Figure 17).

The model implied evolution is given next (Figure 18). Finally, we show the term structure of the implied default probability, both in the real measure $P$ and in the martingale measure $Q$ (Figures 19 and 20).

# 7 Conclusion and Future Work

We sought to implement a reduced model of credit risk to fit Brazilian sovereign term structure, following DPS (2003).

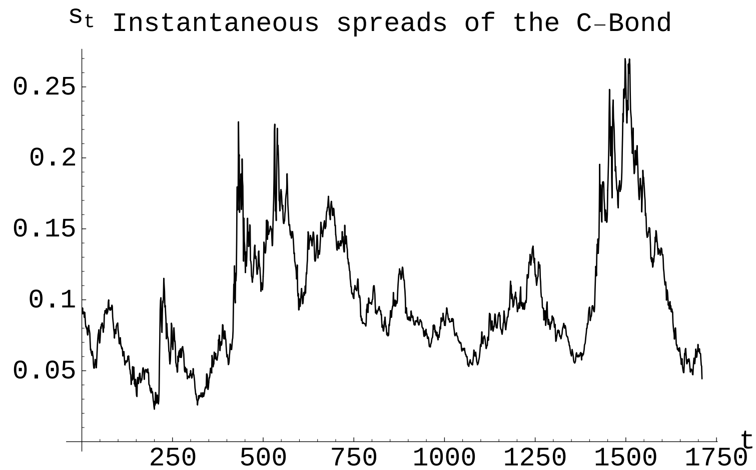



Figure 16: Model implied instantaneous spreads corresponding to the period of the C-Bond data used — 1997-2003.

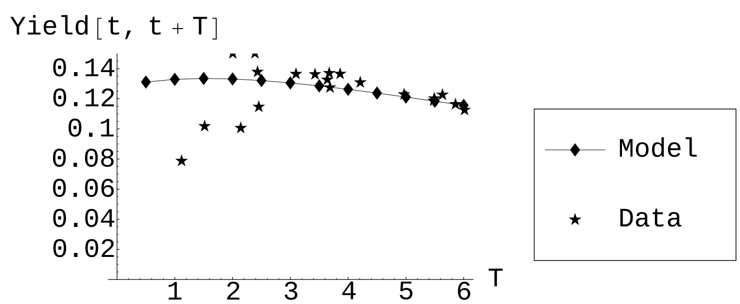



Figure 17: Approximate Brazilian Term Structure of Spreads in January 10, 2003, using data from J.P. Morgan (Spread over duration). Comparison with model implied term structure — January $10^{th}$.

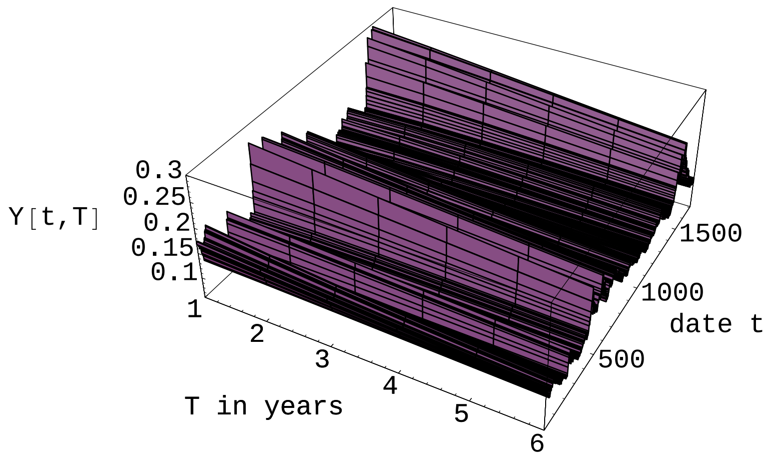



Figure 18: Model Implied Evolution of the Brazilian Sovereign Term Structure.

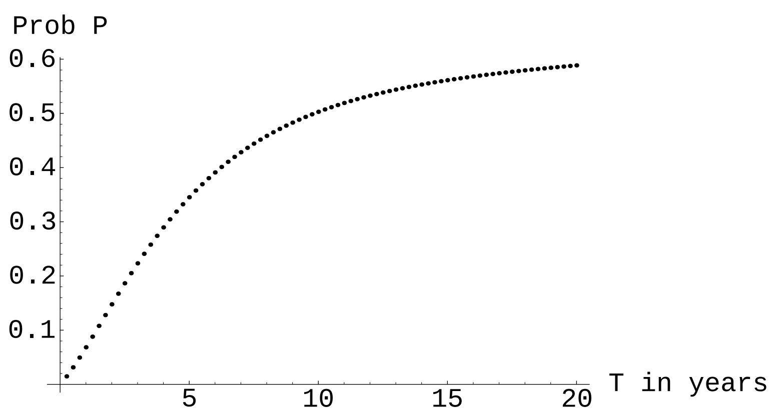



Figure 19: Term Structure of the implied Brazilian default probabilities.

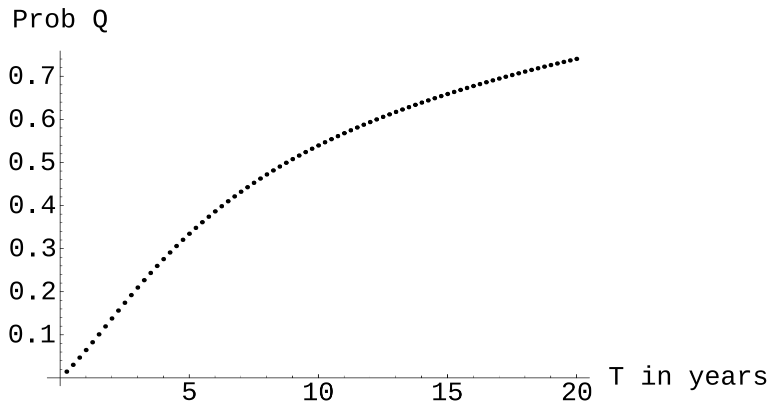



Figure 20: Term Structure of the implied Brazilian default probabilities.

Our results show that the simple model presented needs improvements. We encountered high correlation between some parameters. Not surprisingly, those parameters also generally exhibited large confidence intervals. An adaptation was used to the spread parameters that eliminated high correlation, but it relied on the reference curve parameters, which already showed high correlation. The graphics of the model term structure did not fit real data precisely, although there does exist some resemblance.

However, the results of this paper are not definitive, are part of a broader ongoing research effort of a group at IPEA aiming to study Brazilian term structure. One motivation comes from trying to find the causes of Brazil's very high interest rates.

Besides, quick improvements can be made through better approximations and usage of more sources of data (only one pair of series of yields were used here to make the estimations). If the results do not improve, we will have to implement coupon-bond equations, or construct a zero-coupon yield curve from market data and use this series in the estimation. Furthermore, if those attempts still do not improve substantially the results, we will have to recourse to more advanced models, such as those with stochastic volatility.

Another point for further investigation is the following. DPS considers the variations of the prices of the Russian bonds only with respect to the reference curve and to other Russian bonds. Although we repeated this procedure, we plan to address the question of the correlation among emerging market sovereign bonds, thereby extending the present paper to a 3-factor model. It is a well known fact that a crisis at a specific emerging country can immediately affect the class of all emerging countries.

Related to this is the interesting question of whether the degree of correlation or volatility has augmented since the implementation of the Basel Accord, which regulates the international banks exposition to risky assets via the Value at Risk methodology. That is the result of Zigrand and Danielsson (2001), who study the question of the regulation of risk utilizing general equilibrium, where each agent have to optimize under a VaR (Value at Risk) restriction. They conclude that the regulation in fact diminishes systemic risk, but at the cost of augmenting the volatility of the asset prices.

Another question to be examined is the relation between the spreads due to the credit risk of the sovereign bonds and the prices of the credit derivatives. The credit derivatives market has been developing very rapidly and the Credit Default Swap (CDS) in particular is already a liquid instrument. J.P. Morgan, which maintains the well know EMBI index of country risk, launched in 2003 a new index called EMDI (Emerging Market Derivatives Index) that was following the spreads of the CDS's of 19 of the 31 countries of the EMBI.

Finally, a new line of research combining structural or reduced models and macroeconomics in the sovereign context is expanding. The novelty is to try to use variables with greater economic content or to attempt to construct more realistic structural models using macroeconomic theory to model the debt dynamics, the ultimate source of the credit spread.

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