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The Demand for Money in High Inflation Processes

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THE DEMAND FOR MONEY IN HIGH INFLATION PROCESSES

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1. Introduction

The behavior of the monetary variables under extreme inflation is still a topic of interest and intense research among economists. Recent examples of this are the analysis of inflationary episodes in several countries found in Dornbush and Fischer [1986], Bruno et al [1988], and Dornbush, Sturzenegger and Wolf [1990], and the continuing research on the hyperinflations. The main reason for this interest still is, I believe, the same one pointed by Cagan [1956] in his seminal and now classic study of hyperinflations: these processes provide a unique opportunity to study monetary phenomena.

In hyperinflations the astronomical increases in prices and money dwarf changes in real income and other factors, making it possible to study relations between monetary variables in almost complete isolation of the rest of the economy. In high inflation processes a similar situation occurs, with the high rates of change in nominal variables emphasizing the relationship between money and prices.

At this point it is useful be more specific about is meant by the term "high inflation". It can be recalled that Cagan [1956 p.25] has defined hyperinflation as a process which generates continuously compounded rates of price increase in excess of 50% per month[1], while Dornbush, Sturzenegger and Wolf [1990 p.2] defined extreme inflation as rates above 15% or 20% per month. Dornbush [1992 p.17] considers high inflation an intermediary stage in the process towards extreme inflation, and points out that countries experiencing inflation rates of 10% to 15% per month[2] for any length of time are moving towards hyperinflation[3]. Conceptually, high inflation is a process which, once started, is likely to produce hyperinflation unless it is aborted by stabilization. It also produces important changes in the reaction of agents to inflation, and leads to the creation of mechanisms to offset the effects of inflation like, for example, indexation.

The study of monetary phenomena through the analysis of the dynamics of high inflation episodes has some advantages, when compared to the analysis of hyperinflation dynamics. High inflation processes provide the opportunity to study the effects on money demand of factors which are not relevant in hyperinflations, but are nevertheless of interest and importance, such as the impact of real variables on the monetary sector, and the finer details of the interaction between monetary variables. In addition, high inflation episodes present a larger and more complete database on which

[1]To be precise, Cagan [1956] defined hyperinflations to begin in the month when the monthly rate of price increase exceeded 50%, and as ending in the month before the monthly rise in prices drops below that amount and stays below for at least one year.
[2]All inflation rates in this paper are calculated as monthly continuously compounded rates.
[3]The definition of high inflation is arbitrary, but setting the threshold rate at about 3% per month, is possibly a reasonable choice. The precise definition is not critical for the purposes of this paper.

to make empirical tests, since their occurrence is more common then hyperinflations, which are rather rare phenomena. The main difficulty in modeling money demand in high inflation episodes is that the task must be accomplished in an more complex and noisy environment, when compared with hyperinflations, because the situation resembles less that of a controlled experiment. It is necessary to statistically control for the other variables which affect the demand for money, and deal with a larger and less well-behaved residual in the estimation. Finally, compared with studies of the demand for money under low inflation rates, the analysis under high inflation can benefit from the larger variance displayed by the explanatory variables, allowing sharper identification of the relevant economic factors in the monetary dynamics.

This paper concerns itself with the specification and estimation of the demand for money during high inflation episodes, and proposes a model which can be regarded, in some respects, as an extension of the Cagan [1956] model for the monetary dynamics of hyperinflation. Cagan's main contribution was the stress on real money balances as the relevant dependent variable, and the use of expected inflation as an explanatory variable of money demand, features that have since become part of the standard formulation of the demand for money (see Goldfeld and Sichel [1990]). His model is an adequate starting point for the study of high inflation because it is robust, when applied to the type of processes that he studied. Although many papers have elaborated on several aspects of the Cagan model of hyperinflations, like the identification of the parameters of the equation, the hypotheses regarding the formation of expectations, the functional form of the demand function, the role of foreign exchange, and the estimation procedure, only few of them have rejected Cagan's original model[4]. Only in rare instances did they obtain confidence intervals for the key parameter - the inflation semi-elasticity of the demand for money - which excluded Cagan's original estimates. The model has also been generally corroborated by several recent papers that have taken advantage of the development of cointegration tests to estimate the demand for money for the classic hyperinflations with less stringent requirements regarding the formation of expectations[5].

However, in spite of all the work done on the Cagan model, some of its possible extensions have not been fully exploited in the literature. Some have been hinted upon in his original paper, as for example the possibility that the functional form he employed was too limited, and that in some circumstances real sector variables like income and interest should be included in the regression, while others, like the use of a variable to account for inflation risk, are the result of more recent economic thinking.

[4]See respectively, Barro [1970], Sargent and Wallace [1973], Frankel [1975], Sargent [1977], Frankel [1977], B. Friedman [1978], Frankel [1979], Salemi [1979], Abel et al [1979], Hansen and Sargent [1983].

[5]See Casella [1989], Taylor [1991], Engsted [1993].

These extensions are elaborated here, in the context of high inflation episodes. It is also hoped that the application of the extended model to hyperinflation episodes will contribute to the test of these improvements.

Although the model proposed here is only estimated for the German hyperinflation data, and for the episode of high and extreme inflation in Brazil in the last two decades, I believe that it is of wider applicability. This is so because the features of the money demand equation which is proposed were not obtained in an ad-hoc manner to fit the data at hand, but are derived from theoretical and empirical considerations of a general nature about the demand for money under high inflation, as will be seen below.

The application of the model to the Brazilian experience is of interest in itself, as empirical analysis of money demand during high inflation episodes is not often found in the literature. Montiel [1989] uses the standard money demand specification, but substitutes actual inflation for expected inflation, and obtains equations which are not very satisfactory. Cointegration analysis of money and prices has also been extended to high inflation processes (see, for example, Engsted [1991], Phylaktis and Taylor [1992] and Rossi [1994]), but has been limited by the difficulty of including additional explanatory variables in the equation.

The rest of this paper is divided in 5 sections. The next one derives and presents a model of money demand which emphasizes the discussion of the functional form of the equation, and the role of uncertainty. The third section discusses the empirical specification of the model, with special reference to the formation of expectations. Section 4 presents the estimation of the model for the German hyperinflation data, while section 5 discusses the estimation of the model with Brazilian data for the last two decades. Section 6 summarizes the main results of the paper. The discussion of the estimation procedure and the description of the data for Brazil are relegated to the appendixes.

2. The demand for real balances

In analyzing the demand for money in high inflation processes, one has to consider both the effect of changes in the expected inflation itself, and changes in the variability of inflation. The level of inflation is important because agents will take into account the fact that holding money entails a negative real return, in deciding on their desired money balances. This will lead them to decrease their money demand in response to an increase in the expected rate of inflation, in order to minimize the expected reduction in value of their money holdings. However, the economizing on money balances has a limit because at some level of real balances the benefits of using money as a transactions medium can outweigh the inflationary cost, and induce agents to retain money at rates of inflation at which casual analysis could lead one to expect a total flight from it, as for example, in hyperinflations.

The variability of inflation is also important, since even if expected inflation is constant, agents will also adjust their money balances to compensate for the effect of changes in the inflation risk, since there will exist a component of money demand derived from having to make decisions in an environment with stochastic inflation. This can be justified by discussing two types of effects that an increase in inflation risk can generate in models of the demand for money.

In an inventory-theoretic framework, where there is a penalty for letting real balances decrease below a certain level, or where there is a convenience yield to holding money, agents will maintain money balances which are larger than those held in the non-stochastic situation with the same expected inflation rate. This behavior insures against the possibility that the representative agent will find himself in a situation where his money balances are insufficient for his transactions because of the uncertainty of the inflation rate. The demand for real money balances will be larger, at the expense of consumption, the higher is the probability that a given deviation between actual and expected inflation will occur. These holdings therefore are likely to increase with the variance of inflation, as in Barro [1970], which develops a model similar in spirit to the model of the precautionary demand for money found in Miller and Orr[6] [1966].

On the other hand, in a portfolio-choice framework with risk averse investors, there arises a speculative motif for holding money which, at the margin, induces agents to decrease the demand for real money balances in response to an increase in the uncertainty of inflation, as they try to reduce their holdings of an asset that has become riskier. Therefore, the effect of risk on the speculative demand for real balances has the same direction as that of expected inflation, and opposite to that on the precautionary demand.

If we assume that both effects are in operation, the sign of the coefficient of the variance of inflation in the money demand equation is ambiguous. This is the case in Liviatan and Levhari [1976 and 1977], in a model of money demand with 2 periods, a utility function which depends on the flow of consumption and on the real stock of money balances, a mean-variance terminal utility of wealth function, and a choice set which includes a consumption good and 3 securities: money, index-linked bonds, and nominal bonds. Ambiguity in the sign of the coefficient of the risk variable is also present in Fischer [1975], that assumes the prices of goods and the returns on index bonds, nominal bonds, and equity to be Wiener stochastic processes, and solves the

[6] This effect can be obtained by considering in their model the component of money demand that cannot be explained by other factors besides the uncertainty of the cash flow, and assuming that the stochastic component of the (real) cash flow is due to the variability of inflation.

stochastic dynamic programming problem to compute the asset demands. What I will argue below is that in high inflation situations this ambiguity can be resolved, as it is likely that one of these effects will predominate.

2.1 Money demand in a continuous time consumption and portfolio choice model

The issues raised in the foregoing discussion can be deal with more precisely in following model of money demand, which attempts to illustrate how uncertain inflation may affect the portfolio decisions in high inflation processes. It is structured along the lines of the class of models that originates with Merton's [1969] consumption and portfolio choice model, which has been extended for economies with inflation by Stanley Fischer [1975]. Neither of these models however, deals with money demand. The model here assigns a convenience yield to money and considers it as one of the assets in the economy. It also captures the fact that in high inflation processes the main alternative to holding money may be to hold substitute assets which provide indexation against inflation, which justifies and requires that they be included in the portfolio choice. In many cases this asset is a foreign currency, while in others the private sector may create this alternative asset by indexing bonds to the price of certain commodities as, for example, the *notgueld* in Germany during the hyperinflation. There have been instances where the Government itself created these substitute assets to protect its tax revenue from inflation (the tax pengo in Hungary after the second World War is a classic example), or to be able to increase or maintain its own indebteness (indexed bonds in Brazil, Argentina, and Israel). In certain cases these assets are termed "indexed money", if they acquire high liquidity as their use spreads throughout the economy[7].

Assume prices of the only consumption good follows an Ito diffusion process given by equation (1), where dz is the increment of the Wiener stochastic process[8] z, and where the parameters π and σ are known to the agent and are fixed.

$$\frac{dP}{P} = \pi dt + \sigma dz \tag{1}$$

The model is specified in terms of the real variables, which are equal to the corresponding nominal variable divided by the price level, at each point in time. The agent is endowed with real initial wealth $w(0) = w_0$, at each moment he decides on his

[7]The control of narrow money supply can be very difficult when there exist liquid indexed assets, since transfers of wealth to and from these assets may produce large variations in the money balances. In this situation the monetary authority may be able to control only the broad aggregate: narrow money plus indexed money. I will not analyze this problem here.

[8]See, for example, Merton [1971] or Fischer [1975] for an introduction to the use of the methods of dynamic programming with Ito processes in portfolio selection problems.

real consumption flow (c), and allocates his real wealth (w) between money and indexed (real) bonds in proportions η and $(1-\eta)$ respectively. For simplicity it is assumed there is no labor income.

The indexed bond, which is riskless in the real economy, is the only asset besides money[9]. It has a real non-stochastic return of $r\,dt$, and its nominal return is equal to this real return plus the rate of inflation. Equation (2) displays the stochastic process followed by Q_b, the nominal price of these bonds.

$$\frac{dQ_b}{Q_b} = r\,dt + \frac{dP}{P} = (r+\pi)dt + \sigma dz \qquad (2)$$

Money has a null nominal return, since it's price is equal to unity. The real return on money $(q_m = 1/P)$ is stochastic, since the devaluation of real balances depends on the change of prices, and can then be calculated by Ito's lemma, yielding equation (3).

$$\frac{dq_m}{q_m} = (-\pi + \sigma^2)dt - \sigma dz \qquad (3)$$

However, it is assumed that money has a convenience yield, since it allows the agent to economize on the cost of the transactions required to implement his optimal consumption and portfolio plans. No attempt is made here to derive this feature of money from more basic considerations, but it is clear that if this yield did not exist money would be a dominated security in this economy, and would not be held in positive amounts in the agent's portfolio[10]. The usefulness of money is modeled here by introducing in the flow budget balancing equation of the representative agent an expenditure which reflects the opportunity cost in real resources of holding a fraction of his wealth in the real indexed bond. Multiplied by the marginal utility of wealth, this cost can be thought of as the convenience yield of money foregone by the agent.

This cost per unit time (δ) is assumed to be a decreasing function of the proportion of the agent's wealth which is allocated to money (η), to be null when all of his wealth is completely liquid $(\eta = 1)$, and to be infinitely high when the fraction of real money balances in his portfolio approaches zero $(\eta \rightarrow 0)$. It is assumed here that it can be adequately approximated by the negative of a logarithmic function in the interval $0 < \eta < 1$, as shown in equation (4), where $\kappa > 0$ is a parameter whose unit is

[9]This entails no loss of generality, since Fischer [1975] has shown, in a similar model which allows the existence of nominal bonds, that they will be priced at precisely the rate which insures that none of them exist in equilibrium, as long as expectations are homogeneous (see p. 520).

[10]An example of a model where the demand for money depends on expected cost per unit time of meeting the required transactions, which in turn is a function of the variability of the inflation rate is Barro [1970] (see his equation (54)).

that of the real consumption good. Larger values of κ are associated with greater usefulness of money in facilitating transactions, relative to index bonds, and therefore with a larger convenience yield. As κ increases, the cost associated with maintaining a certain fraction of wealth in index bonds increases, an effect which corresponds to a decrease in liquidity of the index bonds.

$$\delta = -\kappa \log(\eta) \qquad \kappa > 0 \tag{4}$$

The optimal control problem to be solved by the representative agent is to maximize the expected discounted utility of the consumption flow (5), subject to a (flow) budget constraint (6), to an initial condition on wealth, and to the transversality condition (7). The state variable is wealth and the controls are consumption and relative money holdings (c and η):

$$\max_{c,\eta} \quad E_0 \int_0^{\infty} e^{-\rho t}\, U(c(t))\, dt \qquad \text{subject to:} \tag{5}$$

$$dw = (1-\eta) w r\, dt + \eta w [(-\pi + \sigma^2)\, dt - \sigma\, dz] + \kappa \log(\eta) dt - c\, dt \tag{6}$$

$$w(0) = w \qquad \text{and} \qquad \lim_{t \to \infty} E\left[e^{-\rho t} V(w(t))\right] = 0 \tag{7}$$

Utility is assumed to depend only on the consumption flow, and money holdings affect utility because of their effect on wealth, which occurs through the cost component (δ). If the indirect utility of wealth function is denoted by $V(w)$, the basic equation of the stochastic control problem stated above is displayed in equation (8).

$$\begin{aligned} \rho V(w) = \max_{c,\eta} \Big\{ U(c) + \big[(1-\eta) r w + \eta(-\pi + \sigma^2) w + \kappa \log(\eta) - c\big] V'(w) + \\ + (1/2)\eta^2 \sigma^2 w^2 V''(w) \Big\} \end{aligned} \tag{8}$$

Finding the first order conditions for the maximization problem inside the brackets in (8), equations (9) and (10) are derived:

$$U'(c) = -V'(w) \tag{9}$$

$$\kappa + (-\pi + \sigma^2 - r) w \eta + w^2 \sigma^2 \frac{V''(w)}{V'(w)} \eta^2 = 0 \tag{10}$$

Equation (10) is quadratic in η and can be rewritten as $\kappa + b\eta + a\eta^2 = 0$, if a and b are defined to represent the corresponding terms in (10). The general solution of this equation is $\eta = (1/2a)\left(-b \pm \sqrt{b^2 - 4a\kappa}\right)$ but, if $4a\kappa << b^2$, a Taylor expansion of the square root in the neighborhood of b^2, followed by the appropriate simplifications, will produce roots $\eta_1 = -\kappa/b$ and $\eta_2 = -b/a + \kappa/b$. Since in our case $a < 0$ and $b < 0$, the positive root is η_1. Using the expression for b, this solution to equation (10) is shown in equation (11).

$$\eta = \frac{\kappa}{w(\pi - \sigma^2 + r)} \tag{11}$$

In the analysis of money demand in high inflation processes the approximation used above is justified, since in those cases the required condition for it to be valid is likely to be satisfied i.e. $(-\pi + \sigma^2 - r)^2 >> 4\kappa\sigma^2(V''/V')$. To show this, first use (11), to approximate κ as $\eta w(\pi - \sigma^2 + r)$, and denote the relative risk aversion coefficient by $A = wV''(w)/V'(w)$, so (12) is equivalent to this last condition. I will argue below that (12) is likely to be satisfied in high inflation processes, since for reasonable values for the parameters involved the order of magnitude of left hand side will be larger than that of the right hand side.

$$(\pi - \sigma^2 + r) >> 4\,\eta\sigma^2 A \tag{12}$$

Define $\vartheta(X)$, the order of magnitude of any variable X, as the integer n such that $10^{n-1} < E(X) < 10^n$. Recall that in high inflation processes $\pi > 0.1$ (on a monthly basis), so that $\vartheta(\pi) \geq 0$, and note also that for price processes which are not too erratic it is likely that $\vartheta(\sigma^2) \leq -1$. If monthly real interest rates are not absurd, $\vartheta(r) \leq -1$. Therefore, $\vartheta(\pi - \sigma^2 + r) \geq 0$. Now note that κ cannot be large enough (relative to total wealth) to make holding money so desirable that η will depart much from zero[11], since one of the characteristics of high inflation processes is precisely the flight from money, which corresponds to a small share of money balances in the agents' portfolio (say, $\vartheta(\eta) = -1$). To find bounds for the coefficient of relative risk aversion, one can use the results of Chechetti, Lam and Mark [1994], that estimate a value for A of approximately 6 using annual data on equity and bonds returns in the United States over the last century, of approximately 2 for data on monthly stock prices and Treasury debt, a value of 20 for monthly Treasury bill term structure data, and of 15 for data on the returns on five foreign currencies, so that if agents are not "too" risk averse, it is probably reasonable to take $\vartheta(A) = 1$. Therefore,

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[11] Note that the solution for η shown in equation (11) is also the solution of a modified equation (10), where the quadratic term is dropped. This simplification would be reasonable as long as $\eta \approx 0$.

$\vartheta(4\eta\sigma^2 A) = -1$. In conclusion, the comparison of the orders of magnitude of the left and right hand sides of (12) shows that the approximation used to derive (11) is likely to be usable, as long as the variance of the stochastic process of prices is not too large, money is not so indispensable as to make its share of the agent's portfolio be high, and that agents are not excessively risk averse.

This last remark is important in understanding, from the economic point-of-view, the response of this approximate solution to the stochastic programming problem (5)-(7), to an increase in the inflation risk. As the variance of the inflation rate increases there is an *increase* in the demand for money balances, to insure against the possibility of having to conduct business with insufficient stocks of money, and having to forego the high convenience yields of money. This is the effect which is captured by equation (11). There may also be other effect, which tends to *reduce* money demand: the increase in the variance of the inflation rate would increase the risk of changes in wealth due to holding stocks of money, and would induce risk averse agents to reduce their money stocks, in order to reduce the risk of the total portfolio. This effect is not present in our money demand equation, because it is important only if the degree of aversion to risk is large enough to produce a significant adjustment in money holdings. It will only be large if the benefits of avoiding the added risk are large enough justify incurring the larger costs of conducting business with (significantly) smaller money holdings. This portfolio motif is the one which is implicitly assumed to be of one order of magnitude smaller than the transactions motif. Hence, the importance that the risk aversion coefficient not be too high for the approximation to be valid.

Since we are basically interested in the demand for real balances, which equals ηw and is already specified in terms of the parameters in (11), it is unnecessary to proceed with the solution of the control problem, as long as it is assumed that the solution exists. It is not usually the case that one can stop so early, since the first order condition normally involves the indirect utility of wealth function V, which has to be found by solving (8).

The analysis of the approximate real money demand equation obtained above shows that it has the expected signs for the partial derivatives: positive for the variance of inflation and for the convenience yield of money, and negative for inflation and interest rates. It is also interesting to note that if the parameter of convenience yield of money function (κ) is reduced by, for example, the creation of new indexed assets, or by the increase of the liquidity of the existing assets, money demand is reduced in the same proportion.

The functional form of (11) is equivalent to the log-log specification usually employed in the empirical analysis of money demand, and is quite different, specially in terms of implications, from the log linear from employed by Cagan[12]. The issue of the correct functional form to use for the money demand function is dealt with in the next section, in a framework that allows simultaneous consideration of these two functional forms.

2.2 Functional form of the money demand schedule

Cagan [1956] specified his model in log-linear form, but considered the possibility that other functional forms could have been more appropriate, when exploring the reasons why his regressions did not fit well the data points close to the end of the hyperinflations. For those observations actual money demand was larger than that which could be accounted for by his equation. In his words, his regression function would fit the data better if it curved upward on the left (for small money balances and high inflation). This led him to discard the data for last few months on several hyperinflations, and to supply two possible explanations for the problem: that the expectation of monetary reform could justify holding these larger balances, or that demand for money balances did not conform to his equation.

The first explanation has been explored by Flood and Garber [1980], with what I believe are not conclusive results, as can be seen by the nature of the statement that summarizes the results of incorporating the probability of reform in Cagan's equation (Appendix F, emphasis mine). "... it *appears* that the instability of the money demand function at the end of the hyperinflation is *reduced somewhat* by accounting for the probability of reform". Regarding the second explanation, which seems do be discarded by Cagan on the basis of an heuristic reasoning, no systematic exploration of alternative functional forms for the demand for money in hyperinflations can be found in the literature[13].

Here I argue that to model money demand under high inflation it is preferable to use instead the Box-Cox transformation[14] (see Box and Cox [1964]) of the normalized real money balances as the dependent variable. Letting z represent the normalized dependent variable, calculated by dividing the observed values of M/P into

[12]The functional form derived in the text is also different from the exponential form obtained by Barro [1970].

[13]Frankel [1977] does test an equation for the German hyperinflation where the *independent* price variable is transformed by the Box-Cox procedure. The Box-Cox transform of the *dependent* variable seems to me more in line with what Cagan meant by alternative functional forms.

[14] The advantage of this transformation over the power transformation z^{λ} is that it is continuous at $\lambda = 0$.

its geometric mean over the sample period[15], this transformation is shown in equation (13), and illustrated by Figure 1.

$$\Phi(z,\lambda)=\begin{Bmatrix}\dfrac{z^{\lambda}-1}{\lambda} & \lambda\neq 0\\ \log(z) & \lambda=0\end{Bmatrix}\quad\text{and}\quad \lim_{\lambda\to 0}\Phi(z,\lambda)=\log(z) \tag{13}$$

Now, suppose that the Box-Cox transformed dependent variable is written as a linear function of the inflation rate, as in equation (15). It is easy to see that this flexible functional form is capable of representing the two main formulations of interest for us: the inverse function obtained in the last section, and the log-linear specification employed by Cagan. To show the first statement, recall that the convenience yield of money parameter (κ) is positive, and note that in equation (14), which is obtained by applying the Box-Cox transformation with $\lambda=-1$ to both sides of the demand schedule $m=\eta w$ derived from (11), the dependent variable turns out to be a linear function of the explanatory variables. The signs of the partial derivatives, discussed in the last section, are preserved, since the transformation has a positive slope. It is also possible to test if the inverse functional form of equation (11) fits well the data by verifying if -1 is contained in the confidence interval for λ. If we assume all other hypotheses of the model are correct, this turns out to be a test of the hypotheses used to justify the approximation employed to derive our demand equation.

$$\Phi(z,-1)=1-z^{-1}=1-(\pi-\sigma^{2}+r)\kappa^{-1} \tag{14}$$

The proof of the second statement above is straightforward, since the natural log is the special case of (13) when the shape parameter is null. This fact allows us to verify the appropriateness of Cagan's assumptions regarding the functional form of the real money demand schedule by testing for $\lambda=0$ in the linear demand equation estimated with the general Box-Cox functional form.

The foregoing reasoning, regarding the test of the functional form of the demand equation on the basis of the value of λ, is valid as long as κ is constant, as was supposed to be the case in the derivation above. If equation (11) is estimated for a period of accelerating inflation, and if the parameter of the convenience yield of money function decreases in the process as, for example, index bonds acquire higher liquidity, the compound effect of the increase of π and the decrease in κ in the money demand equation will distort the inverse functional relation between real balances and inflation,

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[15] This normalization is convenient in what follows, and for the estimation of λ, as will be seen in the next section.

even if equation (11) is valid[16]. This remark will have important consequences in interpreting the results in the empirical sections of this paper, and for the study of high inflation processes in general, as the increase in inflation is likely to lead to the appearance of more liquid indexed assets[17].

Even if the development of the previous section is not used to derive the demand for money, and therefore not used to provide an indication for the functional form of the equation to be estimated, it is still desirable to use the Box-Cox transform of the normalized real money balances as the dependent variable in estimating a money demand equation. This is argued in the following analysis, which shows that this functional form preserves the desirable properties of the money demand function, if that variable is defined as a linear function of the rate of price increase, and some restrictions are placed on λ. This behavioral function is indicated in equation (15), which obtains when all other variables that affect the demand for real money balances are kept constant, and their total effect is represented by γ. Since the Box-Cox transformation has a positive slope, economic theory will require α to be negative, as will be shown later.

$$\Phi(z,\lambda) = \gamma + \alpha\pi \tag{15}$$

The implications of this specification for the properties of the money demand equation can be explored in more detail by examining equation (16), obtained by using (15) in (13) and solving for the value of money balances (z), for different values of the inflation rate (π):

$$z = \begin{cases} [(1+\lambda\gamma) + \alpha\lambda\pi]^{1/\lambda} & \lambda \neq 0 \\ \exp(\alpha\pi + \gamma) & \lambda = 0 \end{cases} \tag{16}$$

Restrictions must be placed on the values of the parameters of equation (16), in such a way as to force it to satisfy the global properties required from a proper money demand function. This concern with the characteristics of the function over its whole domain is justified by the need for the equation to have the expected behavior both at low and at high inflation rates, in order to be able to track adequately a high inflation process.

The value of the normalized money balances must be well defined for any rate of inflation, and this implies that the term in brackets in (16) must be positive. Using this condition, and requiring the slope of the demand for money balances to be

[16] It is difficult to assess whether the functional form parameter changes due only to changes in κ or not, because the convenience yield of money is not directly observable.
[17] Also see further discussion of this matter in connection with the elasticity equation (19).

negative, it is easy to show that $\alpha < 0$. The condition that $\lambda \leq 0$ can then be derived if we require that in the limit, when inflation increases, the demand for money balances given by equation (16) approaches the horizontal axis from above. Restricting our attention to the case $\lambda < 0$, requiring that the term in brackets in (16) be positive for any π (and in particular for $\pi = 0$), and using the fact that α and λ are both negative, it can be shown that $\gamma < (-1/\lambda)$. In addition, when inflation is null z must be larger than one, because the money demand function is decreasing, and the geometric mean of z is unity. Therefore, inspection of (16) for $\pi = 0$ shows that we must require that $(1+\lambda\gamma)^{1/\lambda} > 1$, which implies $\lambda\gamma < 0$, since $\lambda < 0$. This in turn implies that $\gamma > 0$. The case where $\lambda = 0$ only requires that $\gamma > 0$. The inequalities in (17) summarize the foregoing discussion, and state the conditions required for the Box-Cox linear functional of inflation to be a reasonable equation for the demand for real money balances in high inflation processes.

$$\alpha < 0,\ \lambda \leq 0 \text{ and } 0 < \gamma < -1/\lambda \tag{17}$$

As Figure 2 illustrates, if α and γ are kept constant while λ is varied, money balances calculated from the equation with the Box-Cox transformation for $\lambda < 0$ will be larger than those calculated from the equation with the logarithmic transformation $(\lambda = 0)$. The largest balances are obtained for the inverse function $(\lambda = -1)$. To show this, first note that the several functions in this family meet at inflation rate $\pi = -\gamma/\alpha$, since substitution of this value in (16) yields $z = 1$. This means that these curves are in fact comparable, since they all produce the same value for the inflation rate associated with the average money holdings. For rates of inflation different than the one that characterizes the intersection of the curves, larger absolute values of λ correspond to larger money balances. This can be shown by first noting that the slope of these curves at their intersection is equal to α, and is therefore independent of λ. These demand functions display "high contact" at that point[18].

[18] To compare the performance of the functions in (16) in the estimation of a demand function for a given data set one may prefer that the variation of the parameters λ and γ be constrained in such a way as to force all of the functions to yield the same value for the real money demand when there is no inflation. If $z(0) = z_0$, this would imply that $\gamma = (z_0 - 1)/\lambda$. In this case, (16) becomes $z = (z_0 + \alpha\lambda\pi)^{1/\lambda}$, which cross for $\pi_1 = (z_0 - 1)/\alpha\lambda$. Distinctly from the case when only λ is varied, these curves do not have a common slope at the intersection, and for $\pi < \pi_1$, larger $|\lambda|$ produce *smaller* money balances, while for $\pi > \pi_1$ larger $|\lambda|$ produce larger money balances. It is possible to derive, in this alternative framework, results analogous to the ones in the text, regarding convexity, the behavior of the elasticity, and revenue maximization.

Figure 1
Box-Cox Transform of Money Demand

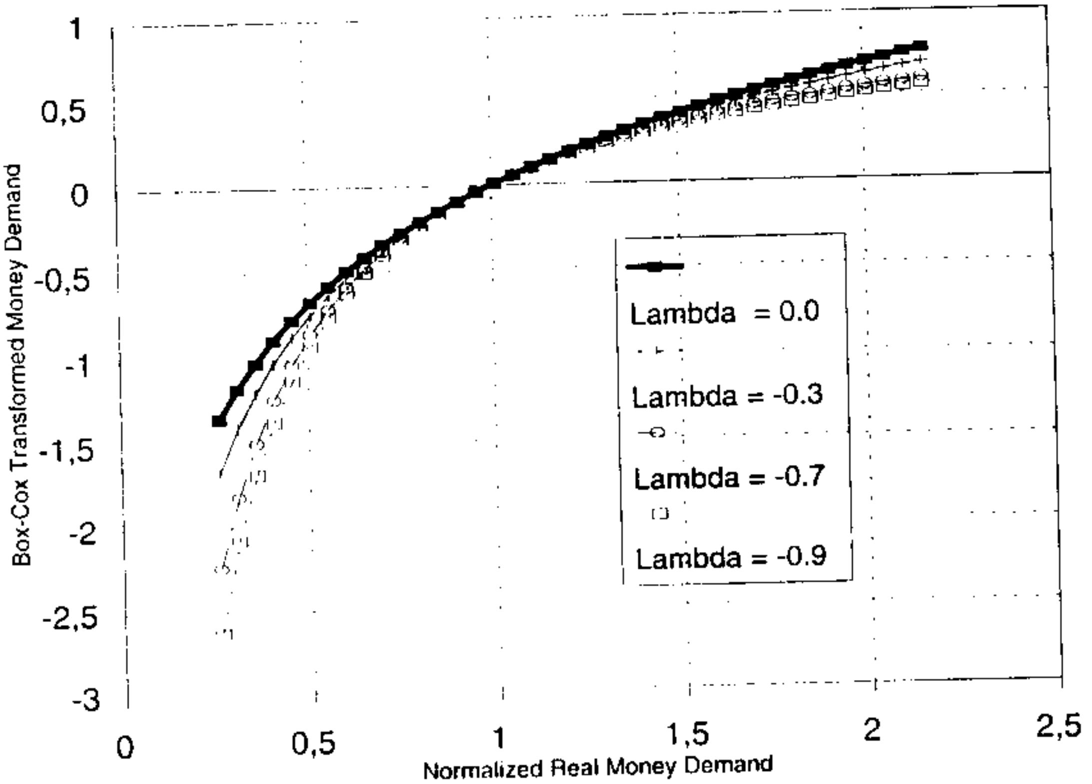
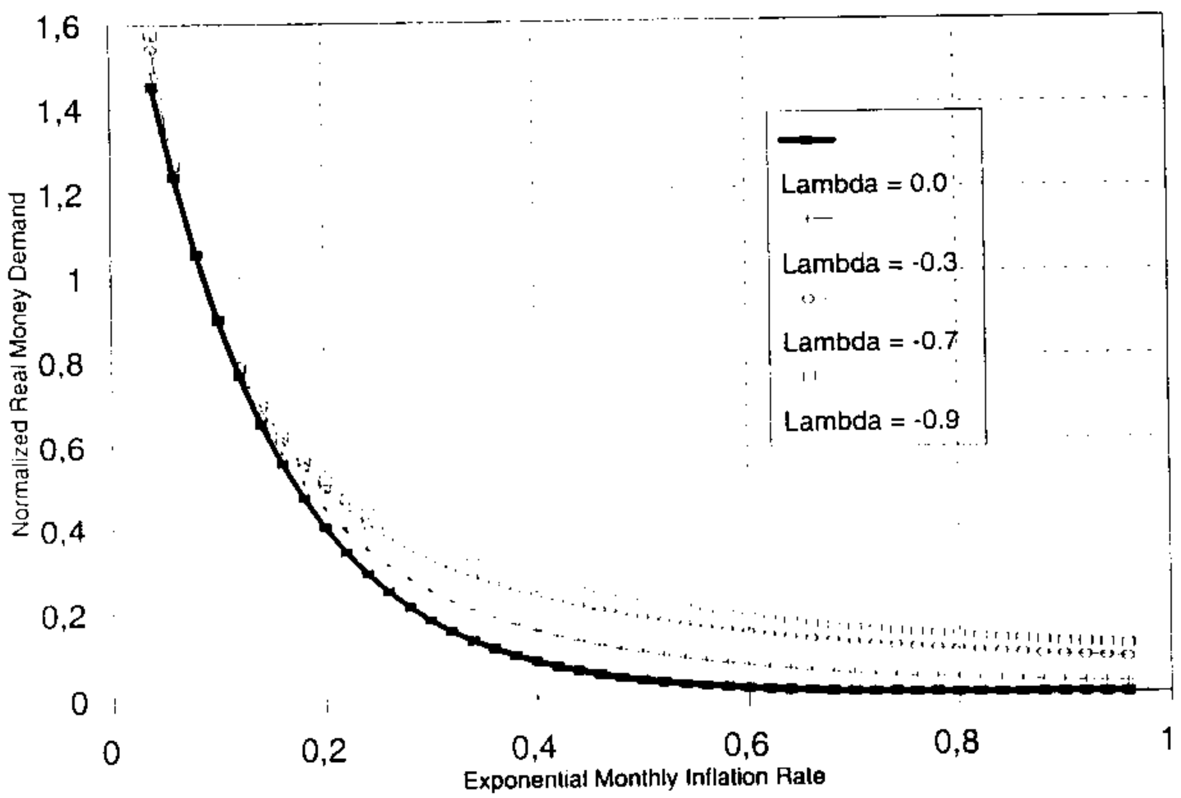


Figure 2
Box-Cox Real Money Demand and Inflation



$$\frac{\mathrm{d}^2 z}{\mathrm{d}\pi^2} = \alpha^2(1-\lambda)\left[1+\lambda\gamma+\alpha\lambda\pi\right]^{(1/\lambda)-2} \tag{18}$$

Since the second derivative of (16) is given by (18), and is positive for all π, it follows that the functions in this family are all convex. The curvature of these demand curves at $\pi = -\gamma/\alpha$ is a decreasing function of λ, since in that case the expression in the right hand side of (18) reduces to the term outside the brackets, where the coefficient of λ is negative. The rate of change of the slope of z is therefore an increasing function of $|\lambda|$, which means that the demand curves which correspond to larger absolute values of the Box-Cox parameter are flatter (steeper), immediately to the right (left) of $\pi = -\gamma/\alpha$. This difference in slope produces the previously stated ranking in the value of money balances in the neighborhood $z = 1$. This ranking of the values of the real money balances with respect to λ extends to all other points of the domain of the functions, because they comprise a continuum and only meet at $z = 1$, as can be seen by inspection of (16). This completes the proof of the assertions regarding the ranking the effect λ on the shape and position of the demand curves.

When the behavior of the elasticity of money balances with respect to inflation (here denoted by φ) is analyzed at high rates of inflation, a crucial difference arises between the two possibilities of equation (16). To see this, we can look at the properties of the elasticity function, which is given by equation (19).

$$\varphi = \frac{\alpha\pi}{1+\lambda\gamma+\lambda\alpha\pi} \tag{19}$$

It can be verified immediately that when $\lambda = 0$ and Φ is the logarithmic transformation, equation (19) reduces to $\varphi = \alpha\pi$, which is the expression for this elasticity derived by Cagan [1956]. This function, however, increases in absolute value without bound as inflation increases, forcing estimated money balances to approach zero very fast as hyperinflation develops, since for any given α their proportionate reduction due to an increase in inflation is very large, and ever increasing.

Alternatively, if $\lambda < 0$, the denominator in equation (19) constrains the growth of the elasticity when π increases, slowing the collapse of money balances, as illustrated by Figure 3. More importantly, the inflation elasticity of the demand for real money balances converges to the inverse of the parameter λ of the Box-Cox transformation when hyperinflation unfolds, as can be seen in (20), which is obtained by applying L'Hospital rule to (19).

$$\lim_{\pi \to \infty} \varphi = \frac{1}{\lambda} \tag{20}$$

Equation (20) shows that when the inflation rate increases without bound, the limiting elasticity for the Box-Cox transformation with $\lambda < 0$ is finite, as opposed to infinite value obtained for the case of $\lambda = 0$. This is consistent with the view that in the former case the speed with which agents reduce their money holdings in hyperinflations is constrained, at the margin, which makes the demand for real balances decline slower in this case. This is the reason why I believe that the difficulty encountered by Cagan in explaining the demand for real money balances near the end of the hyperinflations with the log-linear model can be remedied at least partially by the use of the approach proposed here.

In the other extreme case, when money demand has the inverse functional form of equation (11), i.e. when $\lambda = -1$, the limiting elasticity is equal to -1. This is true as long as the parameter of the convenience yield of money function (κ) is constant. Now consider the possibility that κ may be a function of the inflation rate, with a negative slope ($\kappa' < 0$). The limiting elasticity of the demand for real balances in (11) can easily be calculated as $-1+\zeta$, where $\zeta = \lim(\pi\kappa'/\kappa)$ is the limiting elasticity of the convenience yield parameter when the inflation rate increases without bound[19]. Comparing this expression with the limiting elasticity shown in equation (20), it can be seen that by setting $\lambda = 1/(\zeta - 1)$ it is possible to produce in the general Box-Cox formulation of equation (16) the limiting elasticity of the extended (for non-constant κ) model of equation (11).

If the empirical estimate of λ in the general Box-Cox model responds mainly to the value of the limiting elasticity of money demand at high inflation rates, as will be the case if it is to be instrumental to capture Cagan's larger than expected money balances in hyperinflations, then it is possible that its value may, in fact, be reflecting the behavior of the convenience yield of money at those rates. Decreases in the absolute value of the shape parameter of the Box-Cox formulation, that may occur in estimating the equation for the several stages of a hyper inflationary process, may be due to increases of the absolute value of the elasticity of the parameter of convenience yield of money function (κ), which are a consequence to the appearance of money substitutes. In those cases, therefore, it may be hard to distinguish between two

[19]Formally this extention of the model can be treated as is done in the text (taking the money demand function as fixed) only if it is assumed that agents are unaware of the dependence of the convenience yield parameter on the inflation rate, and take the estimate of the expected κ as given at each point in time, when solving the dynamic programming problem of equations (5)-(7). If the function $\kappa(\pi)$ were known *ex-ante* to the representative agent, it should have been included in the calculation of the optimal money holdings function.

possibilities for the nature of the parameters of the model[20]: flexible λ and κ fixed (equation (16)), or fixed λ and flexible κ (equation (11), extended).

The Box-Cox functional form may also be instrumental in resolving yet another puzzle of the Cagan article: the economies analyzed by him seemed to inflate at a higher rate than the constant rate that would maximize the inflation tax. Since, as seen above, for a given inflation rate the Box-Cox transformation with $\lambda < 0$ will produce a demand for money balances larger than the one found in the case of the log transformation ($\lambda = 0$), it will also produce higher optimal rates for inflation tax maximization, as can be seen in Figure 4. To prove this, note first that if the demand for real balances is given by equation (16), the constant inflation rate that maximizes the inflation tax revenue, which is equal to $\Pi = z\,\pi$, is given by π^* in equation (21).

$$\pi^* = \frac{1+\lambda\gamma}{-\alpha(1+\lambda)} \tag{21}$$

As expected, when $\lambda = 0$ equation (21) reduces to the familiar expression $-1/\alpha$ derived by Cagan for the optimal (for tax collection) inflation rate. Note also that for the inverse function of equation (11) of section 2 ($\lambda \to -1$) the inflation tax maximizing rate will tend to infinity. Therefore, the value of λ is the crucial parameter in determining the behavior of the inflation tax maximizing rate, and has the potential for resolving the apparent divergence encountered by Cagan between the average inflation rate during hyperinflations and the optimal rate calculated from the log-linear model.

Note also from Figure 4 that the revenue curve for the Box-Cox formulation is flatter than for the logarithm specification, which means that errors eventually made in over estimating the optimal inflation rate would produce, in the Box-Cox equation, significantly smaller shortfalls in revenue than in the logarithmic specification. Due to the nature of these revenue curves, which become more skewed as $\lambda \to -1$, the consequence of a given absolute error in setting the inflation rate is less serious if it is made in the direction of inflating "too much", than if it is made in the other direction, of not inflating "enough". This may lead risk-averse decision makers to prefer to err in the direction of over estimating the necessary inflation rate to obtain a given inflation tax revenue, which would lead them to collect more seignorage then needed to absorb a given volume of real resources. One can only conjecture whether this fact will, in a given situation, contribute to the deepening of the hyperinflation process.

[20]To distinguish between these possibilities one could obtain additional data on the convenience yield.

Figure 3
Elasticity of the Box-Cox Money Demand

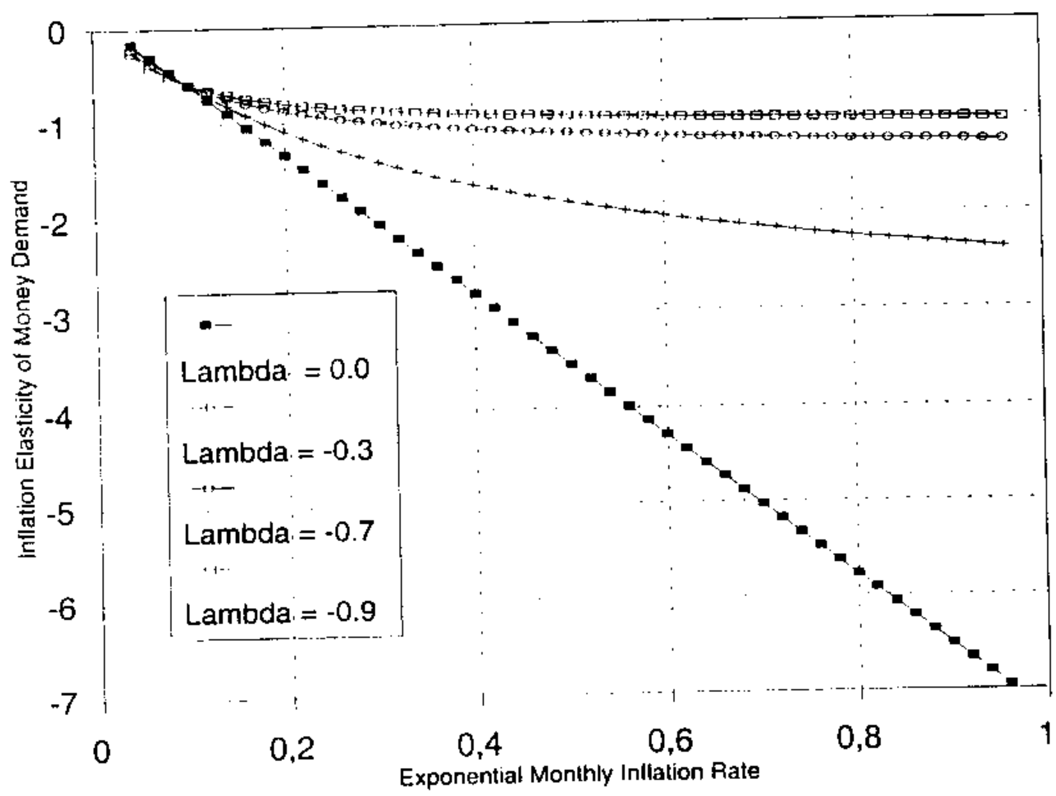
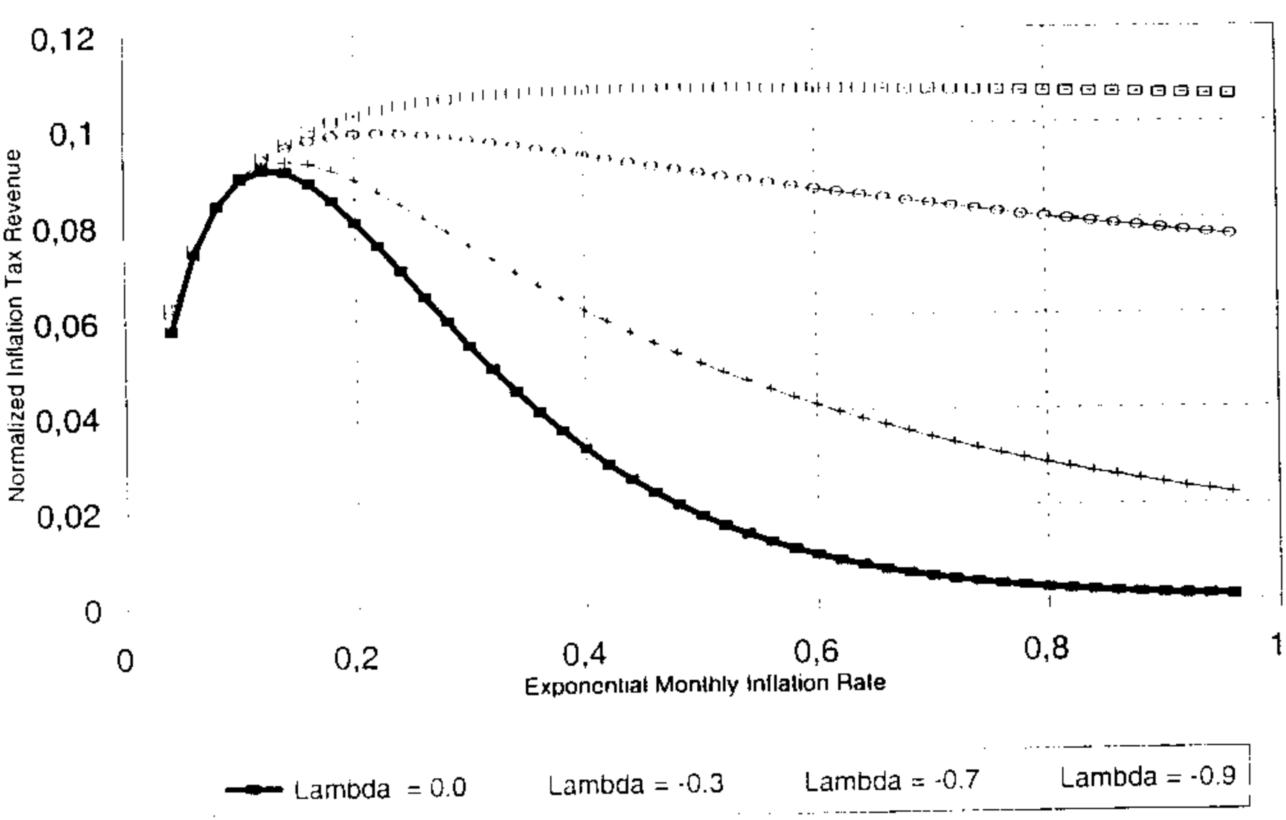


Figure 4
Inflation Revenue: Box-Cox Money Demand



The analysis of (21) also allows us to further restrict the acceptable values for the Box-Cox shape parameter, since for π^* to be positive we must require that $\lambda \geq -1$. Although this condition on λ is not implied by the basic properties of the money demand function, it is desirable that it be satisfied, since it is necessary that the optimal inflation rate for inflation tax collection be well defined, for Cagan's explanation of the reason for the expansion of money balances to be accepted. Aggregating this to the restriction already derived on the Box-Cox parameter, we obtain inequality (22), which can be tested to verify if the adopted functional form is a satisfactory representation for money demand.

$$-1 \leq \lambda \leq 0 \tag{22}$$

The second order condition for a local maximum of the inflation tax revenue at π^* obtains, as long as the conditions already derived for λ and γ are satisfied. To show this, verify that the sign of the second derivative of Π, which is given by (23), is negative at point π^* defined by (21).

$$\frac{d^2\Pi}{d\pi^2} = \alpha\left(1 + \lambda\gamma + \alpha\lambda\pi\right)^{1/\lambda - 1}\left[2 + \frac{\alpha(1-\lambda)\pi}{1 + \lambda\gamma + \alpha\lambda\pi}\right] \tag{23}$$

Note that since the term outside the brackets in (23) is negative, we have to show that the term inside the brackets is positive, at π^*. After some algebra, it is straightforward to see that to prove this amounts to showing that (24) is always satisfied.

$$\alpha\pi^*(1-\lambda) > -2(1+\lambda\gamma) \tag{24}$$

Substituting π^* from (21), and recalling that (17) implies $1 + \lambda\gamma > 0$, (24) reduces to the expression $1 < 2$, which is identically true. This completes the proof that Π is concave at π^*. It turns out that this is the only extremum of the function, and therefore it is also the global maximum.

If the behavior of this maximum is explored when λ is varied, it is easy to see that $\pi^* > -1/\alpha$, as long as $\gamma < 1$. This last condition is a consequence of requiring $\gamma < (-1/\lambda)$ (from (17)) to obtain for $\lambda = -1$, which is the most stringent situation in (22). This means that the money demand obtained with the Box-Cox transform with $\lambda < 0$ will generate inflation tax maximizing rates larger than those of the equation obtained with the logarithmic transformation used by Cagan, as we desired to show. It is also easy to see, by inspection of (25), that if $\gamma < 1$ the inflation tax maximizing rate is an increasing function of the absolute value of λ.

$$\frac{d\pi^*}{d|\lambda|} = \frac{\gamma - 1}{\alpha(1+\lambda)^2} \tag{25}$$

Taken as a whole, these features suggest that the Box-Cox functional form may be helpful in tracking the classic hyperinflations data, if the parameter λ is properly estimated. It is adopted here to estimate the demand for money in high inflation processes because it is likely to perform well when high inflation degenerates into hyperinflation. It is also of interest because it is a flexible functional form, capable of producing the log transformation or the inverse transformation of section 2, whichever is in fact the correct one to use[21].

3. Empirical specification

Here I take the usual approach of analyzing the money demand function in isolation and by single equation techniques, thus begging the questions of identification and simultaneous-equations bias. Dealing with this broader issue would require a careful description of the money supply process, which is outside the scope of this paper. Money supply may also be specific to the country and episode being analyzed, and may not be amenable to the more general treatment of this paper[22].

It is also assumed that desired and actual cash balances are equal. As a consequence, the reasoning provided by the partial adjustment model to justify the use of the lagged dependent variable as an explanatory variable does not apply. The past behavior of the variables influences the equation for the current period only through the adaptive expectations mechanism operating in the independent variables, as described below. The absence of the lagged dependent variable from the right hand side of the money demand equation is a departure from the conventional specification, as it was defined by Goldfeld and Sichel [1990 sections 2 and 4]. I believe that this is an important advantage of this specification because in avoids the econometric problems involved in estimating an equation which uses the lagged dependent variable as an explanatory variable, in the presence of serially correlated errors. Also, as pointed out initially by Cagan [1956], the simultaneous use of adaptive expectations and partial adjustment of actual to desired money balances may also pose serious identification problems.

[21]It is also capable of producing the linear function if $\lambda=1$, but that can be ruled out ex-ante because it would generate negative money balances for high enough inflation rates.
[22]An alternative approach, which is the use of the full-information maximum likelihood estimation proposed by Sargent [1977] for eliminating the potential simultaneous-equations asymptotic bias of Cagan's estimation, is not very encouraging because it yielded "loose" estimates of the slope parameter of the demand schedule for money when applied to the data for the classic hyperinflations.

The main explanatory variable is the expected inflation, as is usual in the analysis of high inflation processes, and as was derived in the model of section 2. Inflation risk will also be important in explaining the monetary dynamics. These will be the crucial factors, when hyperinflation unfolds, since the effects of the other (real) variables will be small in comparison.

3.1 Expectations formation

I will assume that expectations of the future rate of price increase are formed adaptively, as proposed by Cagan [1956] for hyperinflations. In his study, only cursory examination of the time series involved was sufficient to establish that the actual rate of change in prices at any moment did not account for the real money balances at that same moment. Balances seemed to depend also on the actual rates of price change in the past, which led him to postulate that real balances depended on expected inflation, which in turn could be calculated as a weighted average of past rates of change, with weights given by an exponential function. This approach has become the usual formulation for the expectations formation mechanism for high inflation processes, according to Dornbush [1992 p. 24]. He points out that there appears to be significant sluggishness in the initial phases of high inflation, as well as a subsequent acceleration, which suggests exactly such expectations mechanism.

The hypotheses of adaptive expectations states that the expected rate of change in prices is revised per period of time proportionately to the difference between the actual and the expected rate. Letting C_t denote the actual instantaneous rate of change in prices at time t, i.e. it is the discrete sample of the continuous process dP/P, and letting E_t denote the expected inflation rate at the same point in time[23], the adaptive expectation hypotheses implies that E_t can be calculated approximately by equation[24] (26)

$$E_t(\beta_e) = \frac{1-\exp(-\beta_e)}{\exp(\beta_e t)} \sum_{x=-T}^{t} C_x \exp(\beta_e x) \qquad \beta_e \geq 0 \tag{26}$$

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[23] Although the model was specified in continuos time, it is estimated here from a data series which is obtained from a sampling of the continuous processes at discrete intervals (one month). To emphasize this, the new notation is introduced. Hansen and Sargent [1983] have explored the possibility that this aggregation over time could impart significant bias in the estimation of Cagan's model, and concluded that for values of β smaller than 1, in the range of values obtained for the hyperinflations he studied, there is at most a very small asymptotic bias to Cagan's estimate of β.

[24] Since Cagan specified his model in continuous time, the exact formula for the expected inflation is an integral analogous to the summation in the text which is not reproduced here. For the discrete approximation to be valid, the time period T has to be chosen in such a way as to assure that the approximation is sufficiently precise. Cagan shows that for the error to be inferior to 0.05%, T can be calculated as $T = (-1/\beta_e)\ln((1-\exp(-\beta_e))/0.00005)$

The parameter β_e is the coefficient of expectation, and measures the speed of adjustment of inflation expectations. It is positive and its unit is the inverse of the time unit ("per month", if C and E are measured on a monthly basis). A large β_e implies fast adjustment, and produces exponential weights which decline rapidly, as values of actual inflation further in the past are included in the summation. A small β_e implies slower adjustment and smaller weight in the summation for values of actual inflation in the recent past. The smaller is the coefficient of expectation, the more expected inflation is delayed in responding to inflationary shocks. The average length of time by which expectations of price changes lag behind actual changes is measured by $1/\beta_e$. Its unit is time and it indicates the position of the baricenter of the pattern of the exponential weights[25] in equation (26). The first term after the equal sign in equation (26) is the normalization factor, which is equal to the infinite sum of the weights in the summation.

One objection to the use of the error-learning mechanism of adaptive expectations is that it may imply a degree of "irrationality" of economic agents, in that they do not change their forecasting method on the face of systematic forecasting errors. However, as Sargent and Wallace [1973] show, adaptive expectations can be rational in the sense of Muth [1961], if expectations regarding the future growth of the money supply are formed on the assumption that the government is financing a roughly fixed rate of real expenditures by money creation. In their empirical analysis they conclude (p. 342): "Our explanation for the feedback from X to m [inflation to money] tends to confirm the wisdom of Cagan's decision to model expectations by extrapolation of lagged rates of inflation. Such method of forming expectations seems to have been rational.". On the other hand, B. Friedman [1978] has argued that the Sargent and Wallace assumptions are equivalent to requiring that the inflation rate follows a zero-drift random walk process with noise, and points out that the examination of the empirical evidence indicates that stochastic process is implausible as a description of the hyperinflations which Cagan studied. Therefore, this objection to adaptive expectations becomes an empirical issue which will hinge on whether the model fits the data well or not.

A rational expectations version of the Cagan specification, without imposing adaptive expectations but with random velocity shocks, is formally rejected in tests performed by Engsted [1993] for the German hyperinflation. He argues however that there is an element of truth in the model, in that deviations from it are transitory. This

[25] If the data is in a monthly basis, $1/\beta$ measures the average lag in months. For example, if $\beta = 0.20$, the average length of weighting patterns is 5 months i.e. the sum of the normalized weights between periods t-5 and t is equal to 0.5. Also, in that case, approximately 90% of the weight is comprised within the last 12 months.

suggests that the smoothing effect of adaptive expectations may be in operation, in spite of non-rational expectations. Taylor [1991 p.338] also interprets his results as rejecting the rational expectations hyperinflation model for the cases studied by Cagan. However, not requiring rational expectations and estimating the model subject only to stationarity of the forecasting errors, which is satisfied by adaptive expectations if the rate of inflation integrates of order one, the model is supported in several cases. These studies therefore suggest that adaptive - but not necessarily rational - expectations may be a reasonable assumption for models of money demand under high inflation.

Rational expectations is equivalent, in the deterministic case, to perfect myopic foresight. In that case actual and expected inflation rates are equal, and systematic forecasting errors are avoided. Perfect foresight can be viewed as a special case of the adaptive expectations hypotheses, as can be seen by taking the limit in equation (26) when β_e is increases. Therefore, not much is lost in assuming that expectations are adaptive, then estimating the confidence interval for β_e, and finally testing the perfect foresight hypotheses by examining whether it includes large values[26] for β_e. Furthermore, from the empirical point of view, the performance of the money demand equation is not likely to be improved by assuming that the relevant variable is actual inflation. Its use may reduce the explanatory power of the money demand equation, when compared to the adaptive expectations formulation, because one degree of freedom is lost in assuming beforehand that β_e is large.

Tests of relative performance of rational and adaptive expectations hypotheses in present value models performed by Chow [1989] also show that adaptive expectations fit his data better. He also illustrates the well-known fact that incorrectly imposing the assumption of rational expectations on an otherwise correct model can lead to unreasonable estimates of important parameters, and in conclusion argues that adaptive expectations can be a useful working hypotheses in econometric practice. Here I heed this advice and not impose rationality ex-ante on the adaptive expectations model.

Another line of reasoning that can be pursued is to accept models that do not meet the rational expectations requirement, but adopt an improved adaptive expectations hypotheses. Evans and Yarrow [1981] extend the adjustment rule of the basic model to include a second-order term, to correct for the error in the estimate of the time derivative of inflation. They claim that with their error-learning process stable equilibrium exist and have "normal" comparative static properties, as opposed to the perfect foresight rational expectations equilibrium, that in their model possesses "perverse" behavior. Frankel [1975] proposes a model in which agents are assumed to

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[26]As an empirical matter, if data are monthly, a value of 5 for the expectations parameter implies and average length of the weighting pattern of 0.5 months, and a concentration of 99% of the weight in the t-period value

form expectations about the entire path of the price level (and thus about the average long-term rate of inflation) as well as about the short-term rate of inflation, and argues that it yields short-run behavior more consistent with the evidence. The argument however is not empirical, but is based instead on simulations of the dynamic response of the model to shocks. To justify his treatment of expectations, he argues that when information is costly it may be "rational" for agents to use a relatively accurate simple mechanism to form expectations, rather than attempt to (expensively) compute rational expectations paths. For either of the models above, it is an empirical question whether these more elaborate formulations of adaptive expectations would fit the data better. On the other hand, merely improving the expectations hypotheses to force expected inflation to track actual inflation more closely may not lead to a better fitting equation, because in the limit we would have the perfect foresight hypotheses, which does not lead to a superior model, as was discussed above.

Here I use an approach similar to the above, by extending the model with adaptive expectations to allow it to deal more accurately with the uncertainty produced by the variability of inflation. However, the nature of the extension is somewhat different, in that rather then making changes to the expectations mechanism to improve it, while keeping the expected value of the inflation rate as the only link to the demand for money, instead, the risk implicit in using forecasts of inflation obtained by assuming adaptive expectations is explicitly taken into account in the demand function. To achieve this end, it is natural that a term proportional to the expected quadratic error in forecasting inflation be included in the demand for money function, and that it be estimated by adaptive expectations on the basis of the past errors.

It turns out that the strategy described in the last paragraph is equivalent to including the variance of inflation in the demand for money function[27], as suggested by the derivation in section 2. Since its value is not known *ex-ante*, and it may vary with time, the variance (σ^2 of equation (1)) has to be estimated at each point in time on the basis of the past squared deviations of the actual inflation (dP/P) from its expected value (π), rather than from the average over the whole period. If the formation of expectations regarding the value of the variance is also assumed to be adaptive, for consistency with the formation of expectations regarding the inflation rate itself, and allowing for a different expectations parameter (β_v), the expression in equation (27) is obtained.

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[27]On using this variable in the equation, however, it is useful to be aware of a stylized fact noted by Barro [1970]: that the variance of inflation generally depends on the intensity of inflation. The direct effect of an increase of the expected of inflation is to decrease the real balances but, due the effect of inflation on the variance of inflation, an indirect effect may be introduced. This effect however, is not taken into account here, as the explanatory variables are assumed to be independent.

$$V_t(\beta_c \beta_v) = \frac{1-\exp(-\beta_v)}{\exp(\beta_v t)} \sum_{x=-T}^{t} ((C_x - E_x)^2 \exp(\beta_v x) \qquad \beta_d \geq 0 \tag{27}$$

3.2 Other explanatory variables

Equation (29) generalizes Cagan's model by including permanent income and "the" interest rate as explanatory variables, to capture the real sector effects. These are important in high inflation episodes because these processes are likely to last for several years, invalidating the usual assumption made in the study of hyperinflations that these effects are negligible, due to the time frame involved and the size of impact of the other factors.

Permanent income is measured by a weighted average of past income indexes, with exponentially declining weights, following a suggestion first made by Milton Friedman [1956 p. 19] in connection with the specification of demand for money function. He also used this measure of permanent income, among others, in his study of the consumption function (M. Friedman [1957 p. 142]). This is shown in equation (28), where I_x is the index of real actual income in period x.

$$Y_t(\beta_y) = \frac{1-\exp(-\beta_y)}{\exp(\beta_y t)} \sum_{x=-T}^{t} I_x \exp(\beta_y x) \qquad \beta_y \geq 0 \tag{28}$$

The real interest rate is included in the empirical money demand equation, as suggested by equation (11). This is a departure from other studies of money demand under high inflation which use the nominal interest rate as the return on alternative assets. The approach here can also be rationalized by assuming that the Fischer Hypotheses is true, based in part on the fact that Phylaktis and Blake [1993] find strong evidence in favor of the validity of this hypotheses for three high-inflation Latin American countries. Only one interest rate is listed in (29) for simplicity, but more generally, one should include the real return on each of the alternative assets in which wealth can be held. Last, but not least, the use of real interest rates has the advantage of avoiding the multicollinearity of the nominal rates of return with the inflation rate.

To summarize, the general specification of the money demand equation with the features derived in sections 2 and 3 is shown in equation (29), where the variables are dated by t and are represented by capital letters, the parameters are represented by Greek letters, and ε is the error term.

$$\Phi(M_t / P_t, \lambda) = \Gamma + \alpha_e E_t(\beta_e) + \alpha_v V_t(\beta_e, \beta_v) + \alpha_r R_t + \alpha_y Y_t(\beta_y) + \alpha_s S_t + \alpha_t t + \varepsilon_t \tag{29}$$

In the study of the hyperinflations it has been suggested that the exchange rate can play an important role in the demand for money during those episodes. Equation (29) could include the forward premium on the exchange rate as an additional explanatory variable, to measure the expected importance of foreign currency as a substitute for money[28], as suggested by Abel et al [1979]. Alternatively, the real rate of return on holding foreign currency could be used, as recommended by the standard practice of including in the equation the return of all alternative assets, but if it is included in nominal terms care must be taken to control for the possible multicollinearity with the expected inflation rate. It is an empirical matter to verify if the exchange rate variable will be empirically significant in a given high inflation process, in the presence of the other variables of equation (29).

A time trend is also present in the suggested specification because high inflation may last for several years, and it is necessary to capture the effect on the demand for real balances of technical progress represented by the widespread use of computers and electronic transactions. The usual caveats, regarding the use of a time trend variable to capture these effects, apply. Finally, a variable to account for seasonal factors which may be present is also included.

In the next two sections the performance of the empirical specification discussed above is assessed by applying it, respectively, to the German hyperinflation data, and to the analysis of the high inflation episode which occurred in Brazil in the last two decades.

4. Money demand during the German hyperinflation

The main improvements for modeling the money demand schedule in high inflation processes that were suggested in section 2 are: (i) the Box-Cox functional form should be able to fit the data better than the original Cagan equation and, (ii) the variance of the inflation rate should be included as an explanatory variable. They were developed on the basis of several considerations, some of them regarding the behavior of the demand for real balances in hyperinflations. A natural test of these hypotheses is to fit the model of equation (29) to the data of German hyperinflation[29], using expected

[28] The forward premium on forcing exchange may also be used as an indirect measure of expected inflation, as proposed by Frankel [1977 and 1979].

[29] The original data is from Cagan [1956], and the transformed variables are as defined in the last section. The equations were estimated by the procedure described in Appendix A.

inflation and inflation risk as the only explanatory variables, to verify if it performs significantly better than the original Cagan model. This is done in two steps: the first evaluates the contribution of using a more flexible functional form, while the second tests the inclusion of the risk variable.

The result of the first step is presented in Table 1, which shows the coefficients for several pairs of equations, where the odd numbered ones correspond to the logarithm functional form, while the even numbered ones correspond to the general Box-Cox functional form. The pairs of equations differ only on the ending date of the sample period for which they are estimated. These extend the period adopted by Cagan (which ends in July 1923) because it is desired to assess whether the model presented here is able to track better the demand for real balances in the end of the hyperinflation process, which corresponds to data points which he excluded from his equation for Germany.

The comparison of the statistics for the residuals of equations G1 and G2 shows that the Box-Cox functional form allows a superior fit for the sample period adopted by Cagan. As the ending date is extended towards the stabilization month (November of 1923), the advantage of the Box-Cox equation is increased, since the fit of the equation with the logarithm functional form deteriorates significantly, while the Box-Cox formulation is able to display superior adherence, as can be seen from the $\overline{R}^2$ statistics in Table 1. These statistics reflect the ability of the Box-Cox equation to track the data at the end of the hyperinflation, as can be seen by examining Figure 5, which compares the actual balances with the money balances projected by the two equations, for the sample period ending in September 1923 (equations (G5) and (G6)).

Another important empirical advantage of the Box-Cox formulation is that the DW statistics obtained with it are higher than those for the logarithm functional form, which are extremely low. The serial correlation of the residuals in the logarithm formulation is estimated to be around 95%, irrespective of the sample ending date, a fact which may be an indication of missepecification[30] of those equations. For the equations with the Box-Cox formulation, however, the auto correlation of the residuals does not seem to pose a serious problem, because the estimated correlation coefficient is smaller than for the logarithm specification, and the coefficients of the variables in the equation do not change significantly when they are estimated with the Cochrane-Orcutt correction procedure. The implications of the low DW statistics appears to be much more serious for the equations with the logarithm functional form, whose coefficients are unstable with respect to the estimation method.

[30]Barro [1970] had already noted the low DW statistics of Cagan's equations.

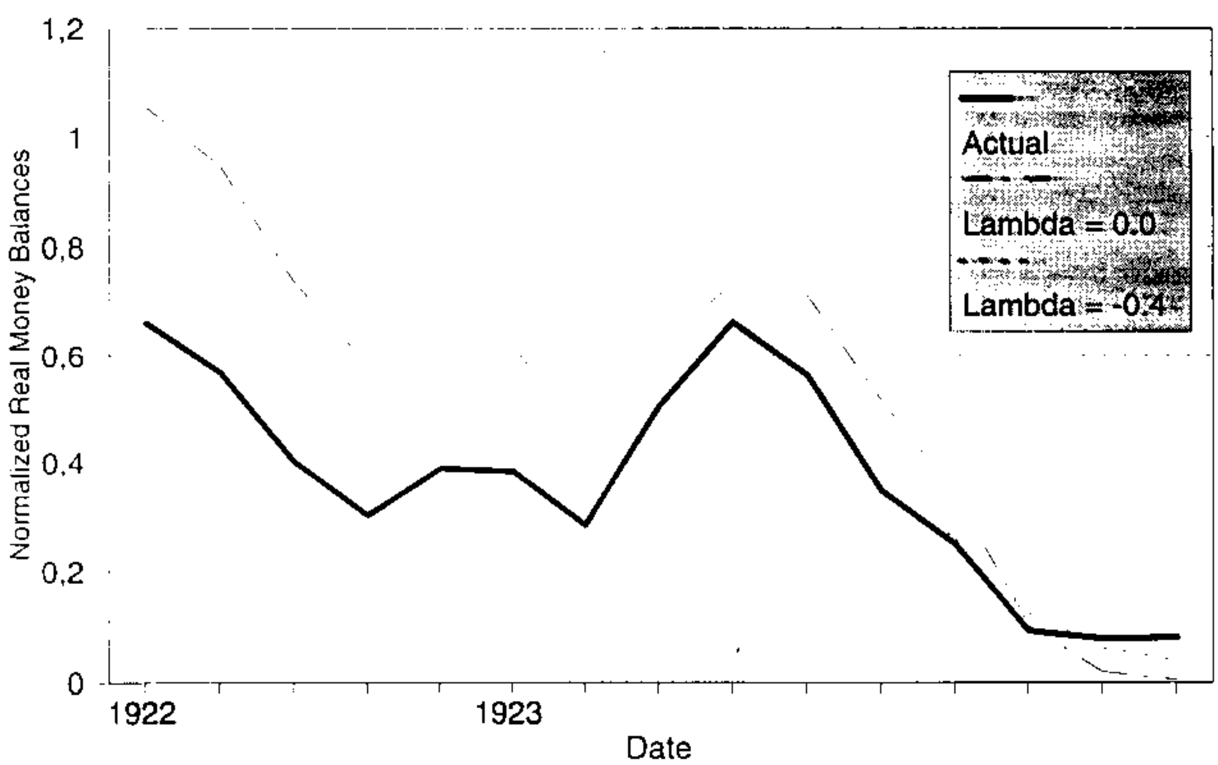
Table 1

Estimates of the Demand for Real Money Balances in the German Hyperinflation[1]
(Box-Cox and logarithm functional forms, without using the inflation risk variable)

Equation code number	G1	G2	G3	G4	G5	G6	G7	G8
Final date	1923:7	1923:7	1923:8	1923:8	1923:9	1923:9	1923:10	1923:10
Deg. freedom	33	32	34	33	35	34	36	35
Functional Form	Log	Box-Cox	Log	Box-Cox	Log	Box-Cox	Log	Box-Cox
R^2	0.9744	0.9870	0.9425	0.9906	0.8572	0.9643	0.8010	0.9123
$\overline{R}^2$	0.9737	0.9867	0.9408	0.9903	0.8531	0.9633	0.7956	0.9099
SSR	0.6824	0.3628	1.927	0.4182	5.822	2.710	9.446	7.107
Durbin-Watson	0.5416	0.5664	0.3775	0.5316	0.4510	1.790	0.3079	0.8347
λ		-0.4		-0.4		-0.5		-0.4
β_e	0.25	0.25	0.2	0.2	0.15	0.15	0.10	0.10
Γ	1.13	1.00	1.06	1.02	0.916	0.963	0.841	0.874
	(0.035)	(0.026)	(0.055)	(0.026)	(0.088)	(0.060)	(0.107)	(0.093)
E	-5.04	-5.20	-4.70	-5.56	-4.08	-5.91	-4.11	-5.73
	(0.14)	(0.10)	(0.20)	(0.093)	(0.28)	(0.19)	(0.34)	(0.29)

[1]The values in parentheses below the variables are standard deviations of the parameters, *conditional* on the values of λ and β_e , *without* correcting for serial correlation of the residuals.

Figure 5
Fitted X Actual Money Demand - Germany



It turns out that for each of the sample periods the optimal parameter for the expectations formation mechanism in the Box-Cox formulation coincides with the one obtained with the logarithm functional form. The point estimate of β_e decreases from 0.25 to 0.10, as the ending sample date is extended from July to October of 1923, but the confidence interval becomes wider, as can be seen in Table 2, which reduces the weight of the evidence in favor of an eventual shift in the value of that parameter. The intersection of the confidence intervals for β_e is (0.17, 0.23), which suggests that the value 0.2, obtained by Cagan, may actually be a reasonable estimate for the true value of that parameter. Also, since their upper limit is 0.28, it is clear that the expected inflation rate is different from the current rate, since the (approximate) equality of these two rates would only obtain for values of the expectations parameter of the order of 10, which suggests that the hypotheses of adaptive expectations was justified.

Table 2

Estimates of the Confidence Intervals for the Parameters of the Box-Cox Real Money Demand Function in the German Hyperinflation[1]

Equation code	G2	G4	G6	G8	G10
λ	(-0.15,-0.55)	(-0.35,-0.45)	(-0.35,-0.65)	(-0.05,-0.65)	(-0.15,-0.45)
β_e	(0.17,0.28)	(0.17,0.28)	(0.05,0.23)	(0.05,0.23)	(0.17,0.23)
β_v					(0.07,0.50)

[1]These 95% confidence intervals were obtained as described in Appendix A, *without* correcting for serial correlation of the residuals. They are also approximate, due to the discrete grid used in the estimation.

The estimated value for the Box-Cox parameter λ in Table 1 is equal to -0.4 for three of the equations, and -0.5 for G6, and is therefore situated near the middle of the interval of possible values defined by inequality (22). The interpretation of this value depends on which of the possibilities discussed in connection with the elasticity calculation in (20) is considered.

First, assuming that the parameter κ, which represents the convenience yield of money, is constant throughout the hyperinflation, it is possible to test the hypotheses that the actual functional form of the demand for real balances is either Cagan's log-linear specification, or the inverse functional form of section 2, by testing for (respectively) $\lambda = 0$ and $\lambda = -1$. The data in Table 2 allows the rejection of both hypotheses for all sample periods considered in Table 1, at the 95% confidence level.

For $\lambda = -0.4$, and for constant κ, the limiting elasticity of real money demand (φ) as the inflation rate increases without bound can be calculated from (20) to be equal to -2.5. It is interesting to compare the value of the elasticity of the demand for real balances implied by the two extreme functional forms for $\pi = 0.5$, which characterizes the beginning of hyperinflation. They are equal to -2.5 and -1.58 respectively, for equations G1 and G2, which shows that at that rate the elasticity for logarithm specification is almost twice that for the Box-Cox equation, and is already equal to the limiting elasticity calculated above.

It is also easy to calculate, using (21) and the parameters of equation G2, that the continuously compounded monthly inflation rate that maximizes the inflation tax revenue is $\pi^* = 19.2\%$, which is not significantly higher then value of 18% calculated by Cagan. If this rate is compared with the continuously compounded average inflation rate[31] for the hyperinflation period (August 1922 to November 1923) of 143%, it must be concluded that the Box-Cox functional form did not help in explaining the divergence between the optimal and the actual rates observed by Cagan [1956]. However, that difference can be put in a better perspective if the optimal rate is compared to the average inflation rate in the period for which the equation is estimated, whose value is between 23.2% and 47.7%, depending on which of the sample periods is considered. The discussion of which is the relevant parameter for comparison with the calculated optimal inflation rate hinges on two aspects: (i) what is assumed to be the occasion of the beginning of the practice of inflationary finance: when inflation started to rise, or only after hyperinflation established itself, and (ii) what is assumed to be the end of the period for which the inflation rate is rationally determined by the attempt to collect seignorage[32]: up to the end of the sample period for which the demand equation is valid, or until the end of the hyperinflation process. If the first alternative is taken to be the answer to these two questions, and July 1923 is taken to be the end of the sample period, the divergence between the optimal and the average inflation rate is rather small.

Now consider the second possibility for the interpretation of the estimated value of λ, which is to assume that the demand function has the inverse functional form, while simultaneously allowing κ vary. In this case the value of the shape parameter of the Box-Cox function is determined by the decreases in the convenience yield of money that occurs in high inflation processes. This was discussed in section 2.2, where it was shown that if the estimate of λ is dominated by the behavior of the money balances at the highest rates of inflation, and if ξ is defined as the limiting

[31]This value was obtained from Cagan's reported value of 322% for the monthly compounded average rate.

[32]It can be argued that after some level of the inflation rate is reached, hyperinflation is the result of fiscal caos, rather than of optimization by policy makers.

elasticity of κ^{*} with respect to inflation, then the following relation between them would hold: $\xi = (1-\lambda)+1$. If that is the case here, then $\lambda = -0.4$ implies $\xi = -1.5$, which is a plausible value.

The second step of the evaluation of the model is to assess the contribution of the inflation risk variable to the equation. For this I use Cagan's sample period, to avoid distortions that may have been introduced by the extension of the sample, and estimate the logarithm and the Box-Cox equations, with the inclusion of the risk variable (V^{*}), to obtain the coefficients shown in Table 3.

Table 3

Estimates of the Demand for Real Money Balances in the German Hyperinflation[1]

(Box-Cox functional form, including the inflation risk variable)

Equation Code Number	Functional Form	R^2 / $\bar{R}^2$	SSR	DW	λ	β_e	β_v	γ	E	V
G9	Log	0.9830 0.9820	0.4524	0.5760		0.20	0.20	1.17 (0.029)	-6.46 (0.276)	5.22 (1.25)
G10	Box-Cox'	0.9907 0.9902	0.2519	0.4677	-0.3	0.20	0.20	1.06 (0.022)	-6.34 (0.206)	4.13 (0.933)

[1]The sample period is September 1920 to July 1923, and the values in parentheses below the variables are standard deviations of the parameters, *conditional* on the values of λ, β_e and β_v, *without* correcting for the serial correlation of the residuals.

They reveal that the fit of the equation is improved, with respect to the equations without the risk variable, as can be seen by comparing equations G9 and G10 of Table 3 with equations G1 and G2 of Table 1, respectively. The shape parameter (λ) increases slightly to -0.3 with the inclusion of the risk variable, but the logarithm functional form is still rejected at the 95% level (see Table 2). The estimate of the parameter of the expectations formation mechanism for the inflation rate (β_e) and for the variance (β_v) are equal to 0.2, which implies a 5 months average lag of, respectively, the expected inflation rate with respect to the actual inflation rate, and of the expected variance with respect to the squared deviations between the expected and the actual inflation rates. The coefficient of the risk variable is significant, since the t-statistic (conditional on the estimated values of λ, β_e and β_d) is equal to 4.4 and 4.2, respectively. Finally, the application of the Cochrane-Orcutt correction procedure does not produce significant changes in the coefficients of equation G10, and yields an estimated correlation coefficient of the residuals of 0.82. For the equation with the logarithm functional form (G9) the situation is not so satisfactory, since the correction for serial correlation of the residuals produces significant changes in the estimated coefficients, and yields an estimated correlation coefficient of 0.97.

Taken as a whole, the analysis of this section shows that the general Box-Cox functional form is better able to track the data of the German hyperinflation than the logarithm functional form[33], and that the inclusion of the variance of the inflation rate as an explanatory variable significantly improves the fit of the equation.

5. Money demand in Brazil 1974-1992

The modeling strategy advanced here was applied to Brazilian data for the period extending from January 1974 to November 1992, when the continuously compounded monthly inflation rate varied from about 3% to close to 60%. After 1986 the were five monetary and price shocks which caused very large changes in the rate of inflation in short periods, generating episodes of steep acceleration and deceleration of prices. The demand for money in the months of very high inflation is about one-tenth (in real terms) that of the beginning of the period (see Figure 6).

The estimation of an adequate equation for the demand for money balances for the whole period specified above turns out to be a challenging exercise, mainly because of the intervening shocks (no data points were excluded). The inclusion of a period of relatively low inflation in the estimation[34] requires the equation to perform well under those situations, and allows us to analyze the changes in the behavior of agents that occur in the initial phases of hyperinflation. On the other hand, the several shocks of the latter part of the estimation period, while difficult to track, are likely to bring out the essential characteristics of the demand for money in this economy.

Econometric estimation of linear models of the demand for money for Brazil which include post-1974 data, and use inflation as one of the explanatory variables, can be found in the following studies. Cardoso [1983] uses quarterly data for the period 1966-I to 1979-IV, in a log-log model, and discusses the relative importance of nominal interest rates and the inflation rate in explaining money demand. The inclusion of lagged money as an explanatory variable in that equation was criticized by Gerlach and De Simone [1985], which estimate an auto regressive distributed lag model on the same variables, but with seasonal factors. They show that the inflation rate is statistically more significant than the nominal interest rate in their equation, contradicting Cardoso's results, and estimate the long-run elasticity of real money demand with respect to inflation to be -0.08. Darrat [1985] also criticizes Cardoso's specification for the same reason, and uses an Almon lag of the logarithm of,

[33]This comparison may not be entirely fair, as the general Box-Cox form has one more degree of freedom, but I believe that the difference is large enough to justify the statement in the text.
[34]From 1974 to 1983 the rate of price increase was below 10% per month.

respectively, income, nominal interest rate, and inflation to resolve the apparent instability in her equation. He also finds inflation to be the main explanatory variable in his money demand equation, and estimates the corresponding long-run elasticity to be equal to -0.20. Rossi [1988] extends Cardoso's data and equations for the period from 1980-I to 1985-IV, and argues that a downward shift in the equation occurred around 1980. He also uses the log-log functional form and finds the equation to be unstable after 1980. None of these models deal explicitly with the problem of expectations formation, nor use alternative functional forms besides the log-log specification, or attempt to include an inflation risk variable in the equation.

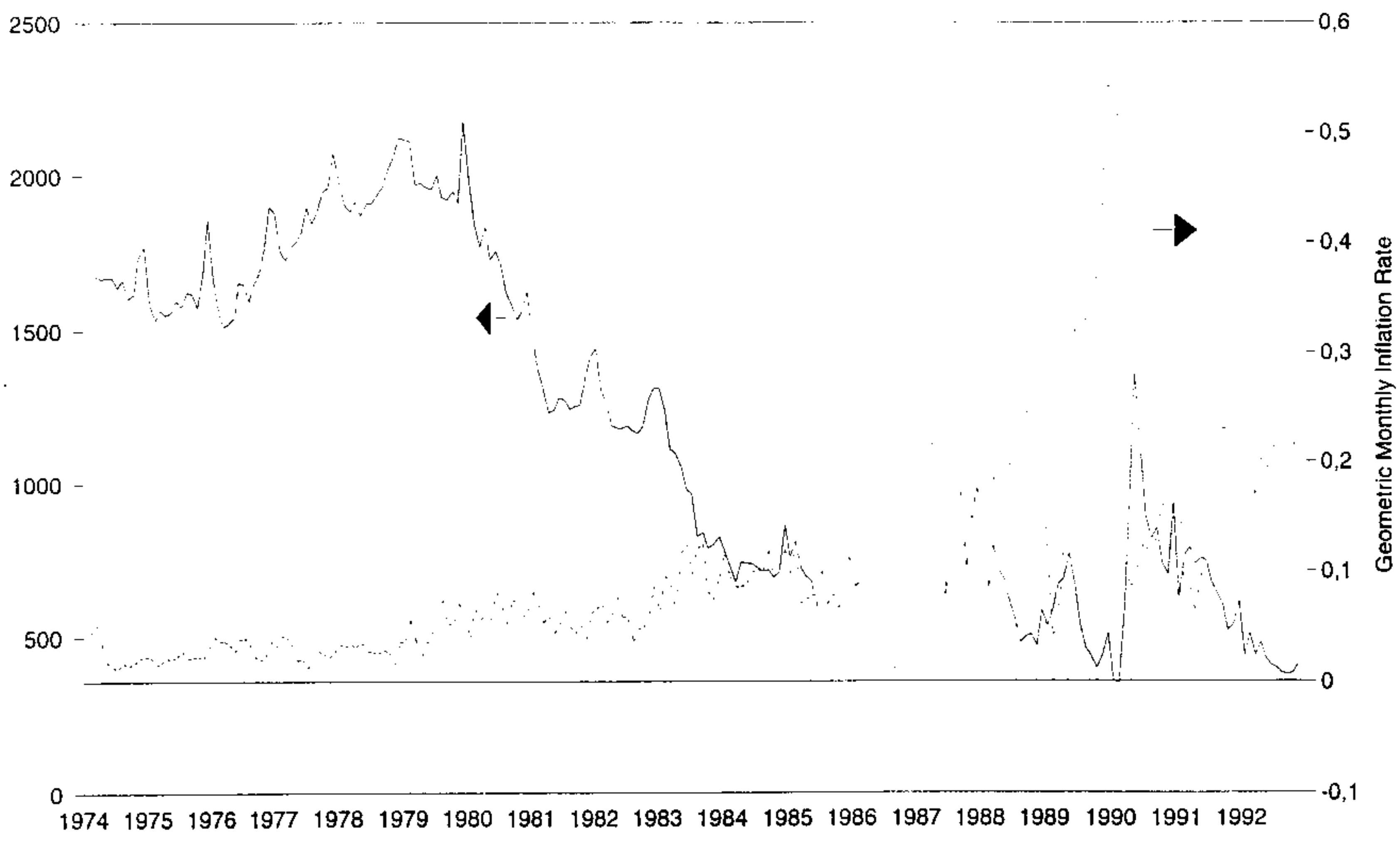
Rossi [1994] applies the methodology proposed by Phylaktis and Taylor [1992] and by Engsted [1993] to test cointegration of money and prices, with monthly data, for the period 1980-1 to 1993-12. It is the only study to include in the analysis the monetary and price shocks that were imposed on the economy to attempt stabilization after 1985. He finds cointegration for the whole period, but not for any of the sub-periods in which he divides his sample. In his log-linear formulation, he estimates the inflation semi-elasticity of money demand to be between -5 and -8 for periods of high inflation, and double those values for periods of moderate inflation.

The model of section 3 suffered some minor adaptations when applied to Brazil. First, the logarithm of expected income is used, rather than the expected income itself, to require the demand for real balances to display constant income elasticity, a feature which seems to be desirable (see Goldfeld and Sichel [1990]). With this income index this property holds exactly when the functional form of (29) reduces to the log specification ($\lambda = 0$), but only approximately in the other cases. As an empirical matter, there is very little difference between equations with and without the log transformation in the income variable for this body of data.

The seasonal adjustment which was deemed useful in estimating the equation for Brazil was the inclusion of a dummy variable for the periods corresponding to the December months, necessary because of spikes which were observed in the real money demand, and are clearly visible in Figure 6. This extraordinary demand in that month is mainly due to the payment to workers of an year-end bonus equal to one month's wages, which is required by law in Brazil. This dummy variable enters equation (29) multiplied by the expected inflation, producing a shift in the slope of the function in that month, instead of shifting the intercept. This reflects approximately a constant proportional increase, rather than a constant absolute increase, of the demand for real balances in December, which seems to be the expected outcome of the increase in nominal incomes which occurs in that month. The technological trend variable is represented by a time displacement variable, equal to the number of months between the initial date and period t.

Figure 6

Brazil - Money Base and Inflation



5.1 Estimation

The adapted model was estimated for Brazilian monthly data for the period 1974:2 to 1992:11, which is presented in Appendix B (see Figure 6 for a graph of the time series of money and inflation). Two different concepts of money[35] were used: the monetary base (MB), equal the outstanding balance of currency issued plus bank reserves (at the last day of the month), and M1, which includes the balance of bank deposits (also at the end of the month). Only the coefficients estimated for the narrower concept are shown here, since the results are similar for the two aggregates, and the conclusions of the empirical analysis do not depend much on which concept is preferred[36].

The coefficients for the six main equations that were estimated[37] are displayed in Table 4, while the confidence intervals for the shape and expectations parameters are shown in Table 5. The equations are segregated into 3 pairs, according to the period covered by each: the complete period (equations B1 and B2), the period before 1985:1 (equations B3 and B4), and after 1985:1 (equations B5 and B6). The equations for the sub-periods were estimated to evaluate the structural stability of the model, and to try to identify behavioral shifts that may occur in the route to hyperinflation. The first sub-period corresponds to the initial phase of the high inflation episode, when the (continuously compounded) monthly inflation rate is below 12% (except for a few months in 1983), while the second is the period of extreme inflation, which starts with an acceleration of inflation during 1985, leading up to the first stabilization attempt in February of 1986. Four other subsequent attempts to control inflation in 1987, 1989, 1990 and 1991, which can be identified as spikes in the inflation rate graph in Figure 6, also failed.

For each of the periods, two estimates of the coefficients were obtained: for the odd numbered equations the maximum likelihood procedure described in Appendix A was used, while for the even numbered ones the Cochrane-Orcutt correction for auto correlated residuals was used to estimate the corresponding odd numbered equation. It should be noted that this correction is only used in estimating the final equation, after the shape and expectations parameters are determined. They are therefore the same for both equations in each pair.

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[35]Data for the nominal money balances at the end of each month is published by the Central Bank of Brazil.

[36] The coefficient of determination of the equations for M1 is marginally larger, but they also have larger auto correlation of the residuals and slightly worse Durbin-Watson statistics.

[37] The RATS program was used to estimate the model.

Table 4
Maximum Likelihood Estimates of the Demand for Real Money Base in Brazil[1]

Equation code number	B1	B2	B3	B4	B5	B6
Initial date	1974:2	1974:2	1974:2	1974:2	1985:2	1985:2
Final date	1992:11	1992:11	1985:1	1985:1	1992:11	1992:11
Observations	226	225	132	131	94	93
Deg. freedom	219	217	125	123	87	85
Estim. method[2]	LINREG	CORC	LINREG	CORC	LINREG	CORC
R^2	0.9596	0.9750	0.9762	0.9866	0.9062	0.9418
$\bar{R}^2$	0.9585	0.9742	0.9750	0.9859	0.8997	0.9370
SSR	3.0819	1.8970	0.2616	0.1462	1.6933	1.0485
SEE	0.1186	0.0935	0.0457	0.0344	0.1395	0.1111
Durbin-Watson	0.7819	2.2191	0.8391	2.1415	0.8311	2.0840
λ	-0.8	-0.8	-0.9	-0.9	-0.4	-0.4
β_e	0.2	0.2	0.1	0.1	0.15	0.15
β_v	0.2	0.2	0.75	0.75	0.3	0.3
β_v	0.075	0.075	0.1	0.1	0.04	0.04
Γ	-8.83	-8.93	-7.05	-6.85	-14.18	-16.82
	(0.69)	(1.45)	(0.54)	(1.09)	(3.04)	(6.18)
E	-8.36	-8.11	-15.06	-14.69	-9.84	-9.39
	(0.24)	(0.39)	(0.77)	(1.61)	(0.42)	(0.70)
R	-1.64	-1.14	-2.33	-1.78	-0.74*	-0.75
	(0.35)	(0.30)	(0.36)	(0.37)	(0.47)	(0.39)
V	60.10	58.53	50.45	12.81#	47.91	43.68
	(2.18)	(3.74)	(17.25)	(18.81)	(2.53)	(4.11)
$\log(Y)$	2.24	2.27	1.83	1.77	3.32	3.90
	(0.16)	(0.34)	(0.13)	(0.26)	(0.70)	(1.42)
$t/100$	-0.78	-0.80	-0.30	-0.27	-0.37	-0.47*
	(0.05)	(0.10)	(0.08)	(0.16)	(0.16)	(0.30)
S^*E	1.91	1.69	1.97	1.28	1.51	1.21
	(0.23)	(0.14)	(0.27)	(0.16)	(0.30)	(0.19)
ρ		0.63		0.73		0.61
		(0.05)		(0.06)		(0.09)

[1] The standard errors conditional on the estimated values of $\lambda, \beta_e, \beta_v$, and β_y, calculated by the standard regression program, are shown in parentheses below the respective estimate. All non-marked coefficients are significantly different from zero at the 5% level, and those marked with * are significant at the 10% level. The coefficients marked with # are not significantly different from zero.

[2] The equations were estimated according to the procedure described in Appendix A. The estimation method indicated in this row refers to the estimation of the final equation obtained by that procedure.

Comparison of equations B1 and B2, which encompass the whole period, reveals that their coefficients are similar, indicating that the specification is very stable with regard to the effects of auto correlated residuals. The largest difference between them is a 30% decrease in the interest rate coefficient when the correction in applied. The coefficient of determination of both equations is high and, considering this is a time series equation, the estimated auto correlation of the residuals is not very high.

The equations for the sub-periods also display good explanatory power and the same kind of stability with respect to the estimation method as the equations for the whole period. The largest differences between the coefficients of equations B3 and B4, for the initial phase of the high inflation process, occurs for the variance of the inflation rate, which is not, however, a significant variable, and for the interest rate variable, which suffers a 24% reduction when the correction is applied[38]. Equations B5 and B6, for the extreme inflation phase, have very similar coefficients.

The immunity of the equations with respect to the estimation problems generated by large auto correlation of the residuals is encouraging, and is not obtained for less complete models of money demand for this period. Of course, the equations B2, B4 and B6 should be preferred in making statistical inferences, as their estimate of the standard errors of the coefficients is unbiased. It should also be noted that condition (22), which was derived theoretically to characterize the economically meaningful values of the shape parameter λ, is respected for all equations, that almost all coefficients are significant at the 5% level, and that their signs are as expected.

5.2 The equation for the complete sample

The estimated value for the Box-Cox parameter λ in equation B1 is -0.8, with an approximate 95% confidence interval of (-0.95, -0.65). This implies, if we assume that the convenience yield of money (κ^{*}) is constant, that the inverse functional form of equation (11) is barely rejected, since it corresponds to $\lambda = -1$. However, while Cagan's log-linear formulation can be rejected at the 1% level, the inverse functional form cannot. The estimated value of the shape parameter indicates that the true functional form is much closer to the inverse function than to the logarithm function, which corresponds to $\lambda = 0$. If κ^{*} is not constant, the estimated value of the shape parameter λ may be distorted by the elasticity of κ^{*} with respect to inflation, as was argued in section 2.2, and the inverse functional form cannot be rejected on the basis of this equation alone. The following discussion, however, assumes that κ^{*} is constant.

[38]There is also a 35% reduction in the coefficient of the seasonal December variable, which has, I believe, a smaller economic importance than the changes in the other coefficients.

The limiting elasticity of real money demand, as the inflation rate increases without bound, is $\varphi = -1.25$, as can easily be seen by using (20). For $\pi = 0.5$, which is the inflation rate that characterizes the beginning of hyperinflation, that elasticity can obtained[39] from equation (19), and is equal to $\varphi = -1.17$, which is a value already close to the limiting elasticity. This indicates that the deepening of hyperinflation would not, for this money demand function, significantly increase the absolute value of the elasticity from the value which is already observed in the beginning of the process. This implication is completely different from that which would be drawn from Cagan's model, which displays an ever increasing elasticity. The value of the elasticity is also only 38% of -3.02, which is the value obtained by using the value of the semi-elasticity of money demand estimated by Rossi[40] [1994] for Brazil, in Cagan's elasticity formula with $\pi = 0.5$.

It is easy to calculate that for the inflation rate that characterizes the beginning of the high inflation episode ($\pi = 0.03$) the value of the elasticity of real money balances with respect to inflation is $\varphi = -0.59$. This value is close to -0.5, which is the theoretical value calculated by Barro [1970] in his optimal payments period model, with a constant fraction of monetized transactions[41]. It is about twice -0.3, which is the value implied by the log-linear model with Rossi's estimate of the semi-elasticity of money demand.

The elasticity of money balances with respect to the rate of inflation, at the average inflation rate for the period analyzed in Cardoso [1983], is equal[42] to -0.48, which is also larger than the estimates obtained for the same period by Gerlach and De Simone [1985] and Darrat [1985], which are respectively -0.08 and -0.20. Since they use a log-log functional form, the elasticity of their demand function is constant, and

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[39] To calculate the elasticity it is necessary to make hypotheses about the values of the other exogenous variables in equation (29). It was assumed in this exercise that the real interest rate is null, the income index is equal to 120 (which is the average value for the second half of the 80's), that the calculation is being performed for the mid 1980's (therefore that t=120), and that the elasticity it is *not* being calculated for a December month. The coefficients of equation B2 were used to obtain $y = 0.976$ which, together with the values of α, λ and the assumed value of π, are then used in (19).

[40] Rossi [1994] estimates an equation with Cagan's log-linear specification for Brazil, for the period 1980.1 to 1993.12, by cointegration methods, finding a super-consistent estimate of $\alpha = 6.04$.

[41] That model is, I believe, the reasonable analog in Barro's paper to the situation of the initial phase of high inflation portrayed in this last elasticity calculation. To compare the estimate for the limiting elasticity with the value which is implied by Barro's model, it is necessary to use the version in which he considers that when hyperinflation gets under way there will appear substitute assets for money which will reduce the fraction of monetized transactions. There he argues that this will lead the elasticity to increase in absolute value, a finding which is consistent with the behavior of that variable in the model presented here.

[42] The average arithmetic *quarterly* rate of inflation for the period included in Cardoso [1983] is, according to Rossi [1988 footnote 8] equal to 6.7%. The implied continuously compounded *monthly* rate is equal to 2.16%, which, through equation (19), yields the required elasticity.

may therefore be capturing to some extent the average behavior of the true elasticity, which is in fact variable, if the model proposed here is correct.

The evidence of the last three paragraphs lends support to the claim made earlier that the Box-Cox transform may be able to capture more precisely the behavior of the elasticity of real money balances in high inflation processes, because it does not increase without bound (in absolute value) as inflation increases. It is smaller (larger) than that implied by the log-linear formulation for large (small) rates of inflation, as was shown to be the case in section 2.2.

The optimal constant continuously compounded rate of price increase for optimal inflation tax collection, calculated from equation (21) for the coefficients of equation B2, is 13.5% per month. It is interesting to note that this rate is not much different from 19.2%, the rate obtained in the last section for Germany, during the hyperinflation. The rate for Brazil corresponds to an yearly rate of 405%, which is significantly larger than the rate of about 250% calculated by Gianbiagi and Pereira [1990], for an equation estimated for the period 1979:4 to 1988:4, and is similar to the monthly rate of 15.2% calculated by Rossi [1994] from data for the period 1980:1 to 1993:12. The inflation tax maximizing rate obtained here is consistent with the possibility that the episode of high inflation which occurred in Brazil in the last decade was caused by an attempt to increase the collection of inflation tax on real balances[43], since it is similar to the average rate of inflation observed for that period.

Table 5
Estimated Confidence Intervals[1] for the Expectations Parameters
of the Base Money (MB) Demand Function for Brazil: 1974-1992

Equation number	B1	B3	B6
λ	(-0.95, -0.65)	(-1.15, -0.65)	(-0.55, 0.15)
β_e	(0.12, 0.23)	(0.06, 0.18)	(0.13, 0.23)
β_v	(0.17, 0.28)	(0.15, 10.0)	(0.22, 0.35)
β_y	(0.06, 0.12)	(0.06, 0.13)	(0.01, 0.18)

[1]The 95% confidence intervals of the shape and expectations parameters were calculated as described in Appendix A.

[43]Also see further discussion of this point in the next section.

The estimates of the expectations parameters for the inflation level (β_π) and risk (β_v) are equal to 0.2, implying that they lag observed values by an average of 5 months. The value for the adaptive expectations parameter for the level of inflation coincides with the value found by Cagan for his pooled hyperinflations data. On the other hand, the value of 0.075 for the expectations parameter for permanent income (β_y) implies an average lag with respect to the actual income index of 13 months, which is consistent with the idea that permanent income ought to respond rather slowly to shocks.

While the approximate income elasticity is high (2.27), the effect of the time displacement variable, which can be interpreted as the effect of technological innovation, reduces money demand each month by approximately 0.8%. It is possible that some degree of multicollinearity may have distorted the estimated value of this last coefficient, since in the equations for both sub-periods it is smaller (about 0.3%). Finally, it is easy to see that the seasonal December dummy variable reduces the (total) coefficient of expected inflation to approximately -6.4 in those months. Finally, when a variable equal to the real dollar exchange rate is introduced in the equation, it turns out not to be significant, which suggests that foreign exchange did not provide any significant hedging opportunities that were not already available to agents through other instruments in this economy, in this period.

5.3 Equations for the sub-periods

Comparison of equations B2, B4 and B6 in Table 4 and inspection of the confidence intervals of Table 5, shows significant differences between several coefficients for the two sub-periods. Since the estimates of λ and the β's change when different samples are considered, the analysis of the stability of the equation for a structural shift due to the beginning, in 1985:1, of the extreme inflation phase of the high inflation process was done in two stages. First, to evaluate the equality of coefficients of the linear part of the specification in the two sub-periods, a Chow test was performed with the equations estimated with the values of shape and expectations parameters (λ and βs) equal those estimated for the whole period in equation B2. Second, the changes in the values the parameters of the non-linear part of the specification from one sub-period to another were analyzed on the basis of the economic interpretation of their implications.

The Chow test at the 5% level rejects the hypothesis that the coefficients of the linear part of the specification in the two sub-periods are equal, conditional on the values of λ and the β's being those estimated in equation B2. However, the same test at the 1% significance level is unable to reject the stability hypothesis. Inspection of the coefficients shows that the main differences between the two periods occurs for the interest rate and income variables[44]. The same test with the coefficients of these variables constrained to equal the value found for the whole sample achieves a significance level of 11.5%, allowing the acceptance of the stability hypothesis in this case. In particular, the coefficient of the expected inflation variable is quite stable around the value found in equation B2. This suggests that the main differences observed in the equations B4 and B6 are due to the non-linear parameters, which are discussed below.

The most interesting difference between the sub-periods is in the value of the parameter λ, which declines form -0.9 in the first sub-period to -0.4 in the second. Assuming that κ is constant, the logarithmic functional form can be rejected for the first sub-period, but the inverse functional form cannot. The opposite situation occurs for the second sub-period, as the inverse function can be rejected but the logarithmic function cannot. The confidence intervals are disjoint, indicating a clear shift.

If the assumption that κ is constant is relaxed, this change can be explained as probably only a reflection of decreases in the convenience yield of money, rather than as an instability of the functional form of the equation. This possibility was discussed in section 2.2, where it was argued that under certain circumstances the following relation between λ and ξ, the limiting elasticity of κ with respect to inflation, would hold: $\xi = (1/\lambda) + 1$. If that is the case here, the observed change in the shape parameter could be due to a change of ξ from -0.1 to -2.3. This is the type of effect what would be expected to happen, as the hyperinflation process evolves: initially, when inflation is relatively low, κ is unresponsive to inflation, but as inflation increases and the extreme inflation phase starts, the elasticity of κ with respect to inflation increases in absolute value, as substitute assets for money are developed, and the convenience yield of money is reduced. Another piece of evidence suggests that this is the case: the estimated value of λ for the German hyperinflation in the last section is -0.4, the same value found here for the extreme inflation phase of the Brazilian process.

The differences in the expectations parameters for the two sub-periods can also be easily interpreted. The increase of β_{π} from 0.10 to 0.15 reflects the shortening of the average lag in expectations formation when inflation increases, an effect which has been perceived and explained by Cagan [1956 p.68-63]. The cost to agents of not adjusting their expectations fast enough rises steeply as the hyper inflationary process deepens, and this induces the agents to shorten their average expectations lag. The decrease of the point estimate of the inflation risk expectations parameter (β_{v}) when

[44] It is not surprising that these two variables seem to be less stable, since the income effect is difficult to identify in the extreme inflation phase due to the recession that occured in the period, and the interest rate variable may be subject to effects from the money supply process.

equations B4 and B6 are compared is not very revealing, since the confidence interval of the first period includes that of the second period. This is only a consequence of the fact that in the initial stage of the high inflation process, when inflation is low, inflation risk is not very important, as shown by the low coefficient of V in equation B4. As a consequence, the expectations parameter is estimated imprecisely, and the confidence interval wide. Finally, the expectations parameter for permanent income (β_y) does not change significantly between the two sub-periods.

To summarize, there are two important differences between the initial phase and the extreme inflation phase of the high inflation process in Brazil in the last 20 years: the absolute value of the shape parameter of the Box-Cox transform was significantly reduced, probably due to the creation of money substitutes, and the average lag in the formation of expectations of inflation was reduced from 10 months to about 6 months.

The implications of these effects for the elasticity of the demand for real balances with respect to inflation, and for the optimal constant rate of price increase for inflation tax collection are easy to assess. The elasticity of the demand for real balances to inflation, as it increases without bound, is equal to -2.5; and for $\pi = 0.5$ that elasticity[45] equals -1.6. They have both increased substantially, when compared with the values obtained for the equation for the whole period, especially the limiting elasticity. This overall increase in the elasticity is probably a consequence of the structural shift represented by the increase in the liquidity of indexed assets that occurred in the period, which was motivated by the demand by economic agents for protection against the capital losses brought about by inflation. In this case, the optimal monthly inflation rate (for inflation tax collection) is reduced to 8.6%, mainly due to the substantial decrease in the absolute value of the estimated shape parameter of the Box-Cox transform, which reflects the larger absolute value of the elasticity of money demand with respect to inflation. The average monthly inflation rate above 20% in the period certainly appears to be non-optimal, from the perspective of inflation tax collection.

6. Conclusion

A specification for an empirical equation for estimating the demand for real balances in high inflation processes, which extends and generalizes Cagan's celebrated hyperinflation model, has been proposed. It was derived from a theoretical stochastic

[45]This limiting elasticity value is different then the one calculated earlier because it uses the coefficients of equation B6, which was fitted for the extreme inflation phase of this high inflation episode.

dynamic programming model of money demand which captures the concept that the convenience yield of money is sharply reduced in hyperinflations, due to the development of substitute assets. The solution of the model suggested the use of a Box-Cox transform of the real money balances, rather then Cagan's logarithm transform, in estimating the equation. It also required the introduction of the variance of the inflation rate in the equation.

The empirical model was specified with adoption of an adaptive, but not necessarily rational, expectations formation mechanism for the inflation rate, the variance of the inflation rate, and income. The empirical specification was completed with the introduction of the real interest rate, of a variable to capture the effects of technological progress, and of a very simple device to capture the seasonal factor which was felt to be relevant. A maximum likelihood procedure was used to estimate the model.

This money demand model for high inflation processes was tested by applying it to the data for the German hyperinflation, and to the analysis of Brazilian inflationary experience of the last two decades. The results validate the model proposed here, and show that its characteristics are indeed important in understanding the demand for money in high inflation processes. It suggests that the demand for money in the initial phase of the that process has a functional form similar to the inverse function (a Box-Cox shape parameter $\lambda \approx -1.0$), which shifts to an intermediary form between it and the logarithm function ($\lambda \approx -0.4$) when the extreme inflation phase establishes itself. These shifts of the Box-Cox parameter are likely to be only a reflection of changes of the elasticity of the convenience yield of money to the inflation rate, which is small in the low inflation phase, but is relevant in the extreme inflation phase.

Further research is necessary to establish if the performance exhibited by the model in these particular high inflation episodes is also obtained in its application to other cases, in other countries.

Appendix A

The estimation of the Box-Cox adaptive expectations money demand model

Equation (29) and definitions (13), (26), (27), (28) form a non-linear system which can be estimated by the two-stage maximum likelihood procedure described below.

I. Let the vector of expectations parameters be denoted by $B=(\beta_e,\beta_v,\beta_y)$, assume that it is known, and use the method proposed by Box and Cox[46] [1964] to produce maximum likelihood estimates of $A=(\gamma_e,\alpha_e,\alpha_v,\alpha_r,\alpha_y,\alpha_s,\alpha_t)$, the parameters in equation (1), as follows:

(a) Calculate the non-observable variables $E(\beta_e),V(\beta_e,\beta_v)$ and $Y(\beta_y)$ from the observed variables C and I.

(b) Calculate the normalized money balances z by dividing each M/P by the geometric mean of the sample values of the real money balances.

(c) For each value of λ, regress $\Phi(z,\lambda)$ on the vector of the explanatory variables of equation (29), recalling that the non-observable ones were calculated in (a). Compute the residual sum of squares, which is conditional on the vector of expectations parameters, and denote it by $RSS(\lambda)|B$. The regression coefficients provide maximum likelihood estimates for A, conditional on λ and on the assumed values for B, which will be represented by $\hat{A}|(\lambda,B)$.

(d) Choose the value of λ for which $RSS(\lambda)|B$ is minimized. This value, denoted $\hat{\lambda}|B$, is the maximum likelihood estimate of λ, conditional on the assumed values for B.

This procedure maximizes the likelihood function (to a second degree approximation) over a set of values for the shape parameter of the Box-Cox transform. These λ's should be chosen to span the range of possible values, which in our case is given by inequalities (13). Considering that in this procedure precision can only be increased by choosing a large number of candidate values for λ, and that this also increases the cost of the estimate, it may be reasonable to select values between -1 and 0, equally spaced by 0.1.

II. Now consider the fact that the vector B is not known, but that it is possible to estimate it by a maximum likelihood method similar to the one proposed by Cagan [1956] for his single dimensional parameter β:

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[46]A handier reference is Maddala [1977 p. 316].

(e) For each value of B, repeat steps (a) to (d), and calculate $RSS(\hat{\lambda})|B$. The maximum likelihood estimate of B (denoted here by $\hat{B}$) is the one for which this value is minimized.

(f) Calculate the unrestricted maximum likelihood estimate of A as $\hat{A}\,|\,\hat{B}$ in (c), and the unrestricted maximum likelihood estimate of λ as $\hat{\lambda}\,|\,\hat{B}$ in (d).

The vectors B have to vary in the grid generated by all the combinations of the possible values for each of the β's. One can follow Cagan's lead and let each β vary in the interval 0.05 to 0.4, with a step size of 0.05, and from 0.5 to 1 with step size 0.1, a strategy that seems to allow reasonable precision for the estimation of parameters which, after all, generate variables which are not observable. Table A1 displays some statistics for the exponential weights in equations (26), (27) and (28), for selected values of β.

Table A1

Characteristics of the exponential weights of adaptive expectations[1]

β (per month)	9	0.05	0.1	0.2	0.3	0.4	0.5	0.75	1.0	5.0
Value of weight when t=0	0.01	0.05	0.10	0.18	0.26	0.33	0.39	0.53	0.63	0.99
$1/\beta$ = Average lag (months)	100	20	10	5	3.3	2.5	2	1.3	1	0.5

[1]Adapted from Table 4 in Cagan [1956]

The procedure described in items I and II above requires a search over a pre-defined grid in the 4-dimensional space (λ, B), where at each interaction a linear regression is run to estimate $A|(\lambda, B)$. Of course, in subsequent runs the grid can be made finer in the neighbourhood of the optimum.

The conditional standard errors of the regression coefficients $A|(\hat{\lambda}, \hat{B})$ can be obtained from the results of the usual regression programs for the equation with the optimal value for parameters λ and B. The standard error and the confidence interval for λ, or for the components of B, (say, ξ in general) can be obtained by an inversion of the likelihood-ratio test statistic. Suppose we are testing the hypotheses $\xi = \xi_0$, and define $\theta = [RSS(\hat{\xi})/RSS(\xi)]^{n/2}$. The variable $-2\log\theta$ is distributed as chi-squared with one degree of freedom. Therefore, the 95% confidence interval for ξ includes all the values such that equation (A1) is satisfied.

$$n \log RSS(\xi) - n \log RSS(\hat{\xi}) < 3.84 \qquad \text{(A1)}$$

Appendix B

The data for estimation of the model for Brazil

Prices (P) were measured by a consumer price index[47] which is an average of prices collected during a month, and is therefore centered on the middle of the month to which it refers. The instantaneous inflation rate at the *end* of month t was then approximated by the exponential rate of growth of the price index between period t and period $t+1$: $C_t = \ln(P_{t+1}/P_t)$.

The real money base was calculated by dividing the nominal balances outstanding in the last day of the month, as published by the Central Bank of Brazil, by the price index at the last day of the month, estimated as the geometric average of the current month's index and the next month's index: $\sqrt{P_t * P_{t+1}}$.

The real interest rate (R) series was constructed by deflating the series for the average monthly nominal interest rate effective for 1-day ahead open market operations (N). The deflation factor was the average inflation for the month to which the interest rate refers, calculated as the geometric average of the instantaneous inflation rates at the beginning and at the end of the month. The *arithmetic* growth rate of prices (instead of the *exponential* growth rate used to calculate C_t) was adopted to measure the instantaneous inflation rate for calculation of the average monthly inflation used to deflate the nominal interest rate, because it was desirable to maintain consistency with the procedure adopted by the Central Bank of Brazil to calculate its real interest rate series for the recent past. As a consequence, the real interest series is calculated as $R_t = (1+N_t)/\sqrt{P_{t+1}/P_{t-1}} - 1$.

To calculate permanent income, data availability was limited. After 1973, a quarterly series of an index of real income[48] (I) was available, and the value for the quarters was repeated for each of its months. None of the variables was seasonally adjusted. Before 1973 only an yearly series was available, and it was used by repeating its value for each of the months. Equation (28) was then used to calculate the expected income.

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[47]The index used was the IGP-DI published by Fundacao Getulio Vargas.

[48] The series for the quarterly index of real output is produced by Instituto Brasileiro de Geografia e Estatistica (IBGE), the official statistics bureau of the Brazilian government.

Y / M	Real Money Base	Infla-tion Rate	Real Interest Rate	Real Income Index	Y / M	Real Money Base	Inflation Rate	Real Interest Rate	Real Income Index
70					**74**				
1		0.0143		47.32	1		0.0269	-0.0127	61.58
2		0.0141		47.32	2	1739.32	0.0433	-0.0234	61.58
3		0.0185		47.32	3	1673.89	0.0523	-0.0356	61.58
4		0.0046		47.32	4	1667.04	0.0343	-0.0307	76.65
5		0.0136		47.32	5	1671.01	0.0180	-0.0137	76.65
6		0.0223		47.32	6	1670.55	0.0126	-0.0003	76.65
7		0.0175		47.32	7	1637.59	0.0125	0.0035	73.15
8		0.0214		47.32	8	1663.36	0.0173	-0.0049	73.15
9		0.0210		47.32	9	1601.86	0.0145	-0.0022	73.15
10		0.0165		47.32	10	1616.93	0.0167	-0.0044	70.34
11		0.0041		47.32	11	1735.13	0.0211	-0.0044	70.34
12		0.0081		47.32	12	1772.66	0.0229	-0.0048	70.34
71					**75**				
1		0.0160		52.66	1	1581.77	0.0224	-0.0049	63.62
2		0.0157		52.66	2	1533.99	0.0154	-0.0030	63.62
3		0.0232		52.66	3	1564.71	0.0173	-0.0061	63.62
4		0.0152		52.66	4	1549.12	0.0212	-0.0074	80.92
5		0.0186		52.66	5	1557.59	0.0208	-0.0065	80.92
6		0.0219		52.66	6	1597.24	0.0223	-0.0041	80.92
7		0.0143		52.66	7	1577.58	0.0277	-0.0086	75.41
8		0.0106		52.66	8	1623.38	0.0212	-0.0086	75.41
9		0.0140		52.66	9	1618.37	0.0227	-0.0064	75.41
10		0.0104		52.66	10	1574.26	0.0222	-0.0031	73.84
11		0.0103		52.66	11	1680.70	0.0217	0.0013	73.84
12		0.0102		52.66	12	1862.00	0.0300	-0.0061	73.84
72					**76**				
1		0.0167		58.13	1	1664.75	0.0408	-0.0120	71.26
2		0.0197	0.0049	58.13	2	1557.80	0.0360	-0.0129	71.26
3		0.0129	0.0079	58.13	3	1511.74	0.0363	-0.0111	71.26
4		0.0127	0.0115	58.13	4	1523.24	0.0336	-0.0052	87.61
5		0.0094	0.0115	58.13	5	1541.31	0.0266	-0.0045	87.61
6		0.0094	0.0087	58.13	6	1655.27	0.0372	-0.0061	87.61
7		0.0123	0.0067	58.13	7	1647.92	0.0400	-0.0056	79.30
8		0.0152	0.0075	58.13	8	1594.69	0.0345	-0.0096	79.30
9		0.0120	0.0098	58.13	9	1648.71	0.0232	0.0039	79.30
10		0.0089	0.0111	58.13	10	1686.33	0.0190	0.0104	81.44
11		0.0088	0.0122	58.13	11	1767.42	0.0223	0.0087	81.44
12		0.0058	0.0082	58.13	12	1902.41	0.0372	0.0058	81.44
73					**77**				
1		0.0173	0.0056	55.57	1	1879.18	0.0302	0.0002	75.13
2		0.0114	0.0055	55.57	2	1755.37	0.0415	-0.0037	75.13
3		0.0141	0.0041	55.57	3	1726.99	0.0398	-0.0097	75.13
4		0.0139	0.0059	69.35	4	1770.03	0.0352	-0.0114	93.53
5		0.0110	0.0084	69.35	5	1785.96	0.0192	-0.0053	93.53
6		0.0108	0.0087	69.35	6	1821.31	0.0208	0.0105	93.53
7		0.0081	0.0082	66.42	7	1900.92	0.0126	0.0158	84.71
8		0.0106	0.0083	66.42	8	1846.74	0.0172	0.0115	84.71
9		0.0105	0.0060	66.42	9	1886.51	0.0272	0.0122	84.71
10		0.0156	0.0050	67.67	10	1946.96	0.0256	-0.0063	84.52
11		0.0103	0.0068	67.67	11	1962.93	0.0214	-0.0028	84.52
12		0.0127	-0.0008	67.67	12	2076.09	0.0261	0.0175	84.52

Y / M	Real Money Base	Infla-tion Rate	Real Interest Rate	Real Income Index	Y / M	Real Money Base	Infla-tion Rate	Real Interest Rate	Real Income Index
78					**82**				
1	1989.48	0.0338	0.0054	77.35	1	1440.41	0.0657	-0.0052	91.22
2	1907.71	0.0319	-0.0039	77.35	2	1300.18	0.0696	-0.0162	91.22
3	1883.93	0.0332	-0.0032	77.35	3	1262.55	0.0524	0.0025	91.22
4	1915.98	0.0314	-0.0007	94.66	4	1187.22	0.0597	0.0015	101.39
5	1869.82	0.0356	-0.0087	94.66	5	1179.18	0.0769	-0.0086	101.39
6	1910.83	0.0280	-0.0031	94.66	6	1179.07	0.0585	-0.0082	101.39
7	1910.34	0.0265	0.0091	90.93	7	1188.30	0.0565	0.0064	99.81
8	1938.40	0.0252	-0.0033	90.93	8	1171.11	0.0361	0.0273	99.81
9	1965.64	0.0285	0.0086	90.93	9	1162.18	0.0467	0.0244	99.81
10	2021.36	0.0271	0.0023	90.08	10	1186.94	0.0488	0.0231	94.11
11	2061.19	0.0151	0.0124	90.08	11	1275.19	0.0596	0.0244	94.11
12	2125.58	0.0356	0.0196	90.08	12	1310.97	0.0863	0.0109	94.11
79					**83**				
1	2118.69	0.0368	-0.0003	83.30	1	1312.61	0.0634	-0.0072	88.32
2	2113.20	0.0561	-0.0126	83.30	2	1236.95	0.0957	-0.0142	88.32
3	1970.46	0.0373	-0.0083	83.30	3	1110.86	0.0880	-0.0019	88.32
4	1977.80	0.0231	0.0055	101.31	4	1096.93	0.0649	0.0285	97.24
5	1964.24	0.0342	-0.0004	101.31	5	1055.47	0.1160	0.0141	97.24
6	1957.56	0.0426	-0.0110	101.31	6	977.91	0.1248	-0.0119	97.24
7	2001.66	0.0565	-0.0304	94.94	7	966.78	0.0965	-0.0094	96.40
8	1928.66	0.0743	-0.0399	94.94	8	825.22	0.1204	-0.0120	96.40
9	1922.54	0.0509	-0.0277	94.94	9	837.68	0.1245	-0.0343	96.40
10	1950.56	0.0542	-0.0302	94.98	10	788.95	0.0809	-0.0101	93.28
11	1912.63	0.0706	-0.0410	94.98	11	798.01	0.0730	0.0129	93.28
12	2182.31	0.0627	-0.0304	94.98	12	824.53	0.0935	0.0067	93.28
80					**84**				
1	1982.83	0.0401	-0.0221	95.49	1	780.76	0.1156	-0.0067	92.07
2	1845.86	0.0657	-0.0278	95.49	2	729.81	0.0950	0.0093	92.07
3	1770.31	0.0553	-0.0243	95.49	3	678.56	0.0856	0.0166	92.07
4	1834.45	0.0614	-0.0386	103.28	4	742.16	0.0849	0.0110	101.80
5	1728.16	0.0550	-0.0384	103.28	5	738.61	0.0884	0.0068	101.80
6	1756.30	0.0811	-0.0438	103.28	6	736.63	0.0982	0.0039	101.80
7	1707.19	0.0678	-0.0415	102.20	7	724.66	0.1010	0.0142	101.61
8	1621.52	0.0520	-0.0288	102.20	8	711.79	0.0999	0.0022	101.61
9	1583.97	0.0733	-0.0285	102.20	9	717.55	0.1184	0.0032	101.61
10	1535.23	0.0723	-0.0338	99.03	10	693.18	0.0942	0.0150	100.14
11	1559.13	0.0580	-0.0251	99.03	11	708.57	0.1002	0.0059	100.14
12	1625.00	0.0637	-0.0055	99.03	12	864.17	0.1190	-0.0001	100.14
81					**85**				
1	1482.41	0.0813	-0.0207	95.37	1	759.26	0.0968	0.0228	98.43
2	1380.21	0.0706	-0.0289	95.37	2	810.80	0.1196	0.0048	98.43
3	1313.61	0.0531	-0.0127	95.37	3	722.80	0.0697	0.0288	98.43
4	1231.99	0.0599	-0.0124	100.60	4	696.72	0.0749	0.0536	107.66
5	1237.34	0.0437	0.0035	100.60	5	681.91	0.0755	0.0418	107.66
6	1278.55	0.0505	-0.0002	100.60	6	598.92	0.0854	0.0217	107.66
7	1272.90	0.0644	-0.0028	96.07	7	716.88	0.1310	-0.0126	110.93
8	1241.93	0.0496	-0.0000	96.07	8	604.81	0.0874	-0.0188	110.93
9	1252.03	0.0430	0.0085	96.07	9	638.31	0.0866	0.0126	110.93
10	1255.36	0.0513	0.0114	90.97	10	593.46	0.1393	-0.0115	109.98
11	1353.75	0.0373	0.0091	90.97	11	622.86	0.1240	-0.0316	109.98
12	1415.94	0.0616	0.0171	90.97	12	763.91	0.1637	-0.0131	109.98

Y / M	Real Money Base	Inflation Rate	Real Interest Rate	Real Income Index	Y / M	Real Money Base	Inflation Rate	Real Interest Rate	Real Income Index
86					90				
1	663.24	0.1396	0.0025	105.64	1	356.52	0.5404	-0.0243	113.77
2	675.44	0.0538	0.0399	105.64	2	352.31	0.5951	0.0319	113.77
3	896.11	-0.0058	-0.0120	105.64	3	606.45	0.1074	-0.0371	113.77
4	1214.45	0.0032	0.0138	115.84	4	935.59	0.0869	-0.0536	115.84
5	1390.98	0.0053	0.0078	115.84	5	1358.24	0.0863	-0.0310	115.84
6	1529.22	0.0063	0.0083	115.84	6	1111.37	0.1220	-0.0200	115.84
7	1728.04	0.0132	0.0096	119.87	7	895.50	0.1216	0.0077	124.90
8	1809.62	0.0109	0.0134	119.87	8	818.19	0.1108	-0.0072	124.90
9	1837.12	0.0138	0.0168	119.87	9	854.11	0.1324	0.0201	124.90
10	1895.86	0.0242	-0.0003	117.90	10	736.34	0.1609	0.0058	115.52
11	1976.15	0.0729	-0.0248	117.90	11	702.48	0.1524	0.0243	115.52
12	1867.43	0.1136	-0.0389	117.90	12	939.03	0.1817	0.0393	115.52
87					91				
1	1591.97	0.1320	-0.0182	113.89	1	629.13	0.1915	0.0040	105.74
2	1324.20	0.1397	0.0442	113.89	2	764.13	0.0700	-0.0620	105.74
3	1160.49	0.1830	-0.0473	113.89	3	790.25	0.0838	0.0092	105.74
4	1036.27	0.2436	-0.0685	123.76	4	735.41	0.0632	0.0189	124.66
5	806.98	0.2301	-0.0165	123.76	5	754.42	0.0940	0.0127	124.66
6	633.86	0.0892	0.0060	123.76	6	745.52	0.1207	-0.0092	124.66
7	763.71	0.0440	0.0189	120.54	7	678.54	0.1440	-0.0154	127.69
8	890.46	0.0771	0.0171	120.54	8	640.62	0.1501	-0.0007	127.69
9	973.32	0.1057	-0.0144	120.54	9	604.71	0.2299	-0.0094	127.69
10	725.50	0.1351	-0.0294	117.79	10	520.90	0.2292	0.0011	117.63
11	897.50	0.1474	-0.0196	117.79	11	537.21	0.2000	0.0685	117.63
12	988.09	0.1751	-0.0266	117.79	12	615.65	0.2378	0.0539	117.63
88					92				
1	837.65	0.1625	-0.0136	113.88	1	441.57	0.2215	0.0258	110.79
2	660.03	0.1669	0.0036	113.88	2	512.07	0.1881	0.0491	110.79
3	796.22	0.1851	-0.0222	113.88	3	440.20	0.1701	0.0605	110.79
4	713.86	0.1782	0.0029	123.26	4	481.65	0.2025	0.0285	121.88
5	685.14	0.1892	-0.0126	123.26	5	432.90	0.1941	0.0088	121.88
6	615.04	0.1951	-0.0084	123.26	6	409.15	0.1963	0.0224	121.88
7	554.54	0.2062	0.0203	123.35	7	398.70	0.2274	0.0211	121.17
8	481.29	0.2292	-0.0135	123.35	8	384.23	0.2420	-0.0064	121.17
9	500.35	0.2436	-0.0033	123.35	9	379.47	0.2226	0.0120	121.17
10	507.90	0.2466	0.0158	115.19	10	384.00	0.2169	0.0289	116.67
11	469.67	0.2538	-0.0002	115.19	11	410.81	0.2127	0.0196	116.67
12	586.05	0.3116	-0.0182	115.19	12			-0.0021	116.67
89									
1	534.65	0.1116	-0.0050	110.75					
2	591.91	0.0414	0.1018	110.75					
3	669.20	0.0504	0.1502	110.75					
4	688.70	0.1200	0.0241	127.49					
5	771.00	0.2372	-0.0677	127.49					
6	677.00	0.3212	-0.0284	127.49					
7	537.50	0.3110	-0.0295	129.77					
8	461.34	0.3287	-0.0249	129.77					
9	438.15	0.3343	-0.0051	129.77					
10	400.14	0.3665	0.0404	122.93					
11	441.23	0.4014	0.0109	122.93					
12	514.07	0.5417	0.0061	122.93					

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