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ISSN 1415-4765

JEL C13,E43,E52,G12

TEXTO PARA DISCUSSÃO

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SINOPSE

Este texto utiliza um modelo de não arbitragem para estudar a interação entre variáveis macro e a estrutura a termo das taxas de juros (ETTJ), interação que é um elemento crítico para política monetária e para a previsão.

O modelo foi utilizado para analisar a ETTJ de títulos emitidos no mercado doméstico do Brasil e a sua relação com a taxa de câmbio e uma medida de inflação esperada, utilizando dados diários no período 2000-2005. Os modelos foram estimados em duas versões. Uma contínua estimada por máxima verossimilhança e outra discreta estimada por Monte Carlo Markov Chain (MCMC).

Concluímos que: 1) os resultados das duas versões foram qualitativamente, e, em muitos casos, quantitativamente iguais, o que sugere a robustez dos resultados; 2) avaliamos a importância relativa das fontes de determinação das ETTJ, em particular dos choques cambiais, de inflação, e de movimentos autônomos da taxa de juros.

ABSTRACT

We use no arbitrage models with macro variables to study the interaction between the macroeconomy and the yield curve. This interaction is a key element for monetary policy and for forecasting. The model was used to analyze the Brazilian domestic financial market using a daily dataset and two versions of the model, one in continuous-time and estimated by maximum likelihood, and the other in discrete-time and estimated by Monte Carlo Markov Chain (MCMC).

Our objective is threefold: 1) To analyze the determinants of the Brazilian domestic term structure considering nominal shocks; 2) To compare the results of the discrete and the continuous time versions considering adherence, forecasting performance and monetary policy analysis; and 3) To evaluate the effect of restrictions on the transition and pricing equations over the model properties.

Our main results are: 1) results from continuous and discrete versions are qualitatively and in most cases quantitatively equivalent; 2) Monetary Authorities are conservative in Brazil, smoothing short rate fluctuations; 3) inflation shock, or slope shock, depending on the model selected, are the main sources of long run fluctuations of nominal variables; and finally, 4) no arbitrage models showed lower forecasting performance than an unrestricted factor model.

SUMMARY

Macro Factors and the Brazilian Yield Curve with No Arbitrage Models

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May 2006

Abstract

We use no arbitrage models with macro variables to study the interaction between the macroeconomy and the yield curve. This interaction is a key element for monetary policy and for forecasting. The model was used to analyze the Brazilian domestic financial market using a daily dataset and 2 versions of the model, one in continuous-time and estimated by maximum likelihood, and the other in discrete-time and estimated by Monte Carlo Markov Chain.

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1 Introduction

The term structure of interest rates synthesizes agents' perceptions about the future state of the economy. The interaction between that perception and macroeconomic variables is an important element to be taken into account by the Monetary Authorities (MA) for policy decisions and for the purpose

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of forecasting by market participants. Ang and Piazzesi (2003), A&P, discuss that interaction combining the financial and the macroeconometric literature.

In the financial literature, the Affine Term Structure Models (Duffie and Kan, 1996) constitute a very popular class of models, in which the yield and the risk premiums are modelled in continuous time as affine functions of unobserved state variables. However, standard affine models do not contain macroeconomic variables, which means that unobservable factors and forecasts cannot be related to macro shocks.

Macroeconometric models analyze the effect of non-financial variables over the yield curve, and model the dynamics of the rates and of the effects of financial and macro shocks. But do not take into account no arbitrage restrictions among the rates of diverse maturities, which can potentially lead to an overparameterization of the model and a reduction of its forecasting capacity.

A&P's model incorporates macro variables into a discrete time affine model and a MA reaction function to nominal shocks, that is, a Taylor rule. In this way they identify in a more ample way the determinants of the dynamics of the yield curve, besides imposing no arbitrage restrictions among the yield maturities.

Due to the inclusion of the macro variables and to the nonlinear character of the model, the task of the inference of the parameters becomes a particularly arduous one, specially because of the high number of parameters and of the identification problems, that are more complex than those in traditional Vector Autoregression (VAR) cases. The difficulty of the inference motivated Ang, Dong and Piazzesi (2005) to use the Markov Chain Monte Carlo (MCMC) algorithm, a Bayesian approach (see Gamermam, 1997, and Johannes and Polson, 2003), that is less vulnerable to dimensional issues than Maximum Likelihood.

The Brazilian financial market, like other emerging countries, has characteristics that make it different from those of developed countries, such as lower bond liquidity, short term structure (less than 3 years), the greater number of interventions that result in changes of regime and of rules of operation, the existence of credit risk of the public debt, the greater volatility of the prices due to macroeconomic instability, and the vulnerability due to the variations of the exchange rates - variations determined in great part by conditions external to the country.

We adapted A&P's model to analyze the yield curve in Brazil, changing the frequency of observed data, the choice of macro variables and the interpretation of the Taylor rule. We used high frequency data (daily) to compensate for the smaller historical period in which the rules and the regime are stable, and using the relevant macro variables available at that frequency.

In Brazil as well as in other emerging countries, the exchange rate fulfill a fundamental role in the price stabilization. It affects directly the price of the tradable products and indirectly the regulated prices and price expectations. Also, it depends on international markets. Since January 1999, Brazil started to operate under a floating exchange rate regime, which causes exchange rate shocks pass through domestic consumer prices and inflation. Expected inflation and exchange rates are the main macro variables that interact with interest rates, and those variables are available at daily frequency.

The contracts traded at the Brazilian futures exchange, *Bolsa de Mercadoria e Futuros* (BM&F), permit the estimation of the domestic term structure and of the market expectation of the inflation rate for various future horizons. The floating exchange rate regime and the availability of expected inflation data reduced our sample to the period from April 2002 to October 2005 in a total of 870 days.

The numerical problems resulting from the use of high frequency series were dealt with by the specification and estimation of the model in two versions. The version (C) is defined in continuous time and estimated by maximum likelihood, while version (D) was defined in discrete time and estimated with MCMC. After the models are estimated, we defined measures of adjustment, of forecasting capacity and impulse response functions in such a way the main results of the two versions are comparable.

Our choice of also using continuous time modelling was motivated from the tradition of a large body of financial literature. However, the computational burden is far higher than in the discrete time model, which proved to fit our needs even in the high frequency exercise. Furthermore, the use of MCMC algorithms besides the more common maximum likelihood method is intended to deal with the delicate question of assessing the robustness of inference results.

We use time-varying affine risk premium, which improves the fitting of the model, but it should be said that the affine representation has not yet been justified theoretically in terms of underlying preferences.

Ang, Dong and Piazzezi (2006) use quartely data and interprets the relation between short rate and state variables as Taylor Rule, a reaction function of MA to inflation. Since we use daily data, it is not possible to use the same interpretation, MA do not react so quickly. Thus, in our case this equation represents market reaction function, and we will denote it as "Taylor" like rule.

Inspired in their investigation of the comparison of different specifications of the Taylor rule (the Backward-Looking, the Infinite Forward-Looking and the Standard Taylor rules) we test restrictions on this market reaction function. Finally we study other specifications not under the class of no arbitrage models (Factor Models and a modified version of the Diebold and Li (2006) model).

Our objective is threefold: 1) To analyze the determinants of the Brazilian domestic term structure considering nominal shocks; 2) To measure the forecasting performance of the models; 3) To compare the results of the continuous and discrete time models and of maximum likelihood and MCMC inference methods.

The main results are: 1) results from continuous and discrete versions are qualitatively and in most cases quantitatively equivalent; 2) inflation shock, or slope shock, depending on the model select, are the main sources of long run fluctuations of nominal variables; and finally 3) no arbitrage models showed lower forecasting perfomance than non restricted versions.

The articles most closely followed are Ang and Piazzesi (2003) and Ang, Dong and Piazzesi (2005), which first allowed the incorporation of macroeconomic variables as state variables alongside the latent state variables of the traditional term structure models to study public policy effects on the yield curve and vice-versa. They use a discrete time specification of an Affine term structure model which is at the same time a Vector Autoregressive (VAR) model. Also, the no-arbitrage condition is enforced in the model, a restriction not necessarily followed in Macro VAR models, while the flexibility and advantages of the VAR model are retained.

Ang and Piazzesi (2003) estimate via maximum likelihood a macro-to-yield model, in the sense that macro factors affect, but are not affected by, monetary factors. Ang, Dong and Piazzesi improves that model estimating a bidirectional model with one latent factor and two macro factors using MCMC. They report that the no arbitrage VAR models performs better forecasting than the unrestricted VAR. Also, Diebold, Piazzesi and Rudebusch (2005) remark that pure affine no arbitrage models add little insight into the nature of the underlying economic forces driving the yield curve movements and adding macro factors shed light to the fundamental determinants of the interest rates. They survey Macro-Finance models, pointing out the importance of the short rate as the fundamental building block to price all the bonds and as a policy instrument under direct control of the central bank to achieve its economic stabilization goals.

Rudebusch & Wu (2004) develop a no arbitrage macro-structural model with macro variables and latent monetary factors that jointly drive yields.They report that output shocks have a significant impact on intermediate yields and curvature and that inflation surprises have large effects on the level of the yield curve. Another finding is an improved forecasting when macro factors are added to the usual latent factors model, and reasonable interpretations about impulse response of identified shocks. Dai and Philippon estimate a no arbitrage VAR model with one latent factor and government deficit, inflation and real activity as macro variables. They document that deficit is an important factor behind the yield curve. All those papers use discrete time model on monthly or quarterly frequency.

In contrast, we use daily data, two latent variables plus two macro factors, discrete and continuous time specifications, and estimate the parameters using both maximum likelihood and MCMC. Thus, we can directly compare how modelling and inference choices impact the results. Also of note is the fact that we estimate over a more volatile emerging market economy under constraints of time series size.

2 Term Structure Models

Let Y be the vector of the interest rates for the n selected maturities and $X = (M, \theta)$ the vector of the $p < n$ variables that characterize the state of the economy, where θ is the vector of the q unobservable monetary factors and M is the vector of the $p-q$ observed macroeconomic variables. In our models the trajectory of Y is described by the sum of the effects of the state variables $B(.)X$ and of independent errors du. In essence, the function $B(.)$ prices the bonds of various maturities with respect to the instantaneous interest rate r so that, using a risk premium λ that is affine in the state variables, the result is an affine relationship between the yields and the state variables:

$$Y = A(.) + B(r, \lambda, .)X + \sigma du, \; du \sim N(0, I_n) \tag{1}$$

The instantaneous interest rate varies according to the estate of the state of the economy. With daily data we can not interpret this equation as a Taylor rule because Monetary Authority do not react in such a high frequency. In out model, the reaction of Monetary Authority appears only implicitly in the dynamic equation that links latent to macro variables. This have two implications, the coefficients of this equation do not have interpretation, but the impulse response function of the model to inflation shock carries the effect of the Taylor rule.

$$r = \delta_0 + \delta_1 \cdot X + \sigma_1 du_1, \tag{2}$$

the variable premium is an affine function of X,

$$\lambda = \lambda_0 + \lambda_1 \cdot X, \tag{3}$$

and the dynamics of the state variables is described by a mean reversion multivariate model in which the shocks are assumed correlated:

$$dX = \mu + \Phi(\overline{X} - X) + \Sigma de, \; de \sim (0, I_p), \tag{4}$$

Once the parameters $\psi = (\lambda_0, \lambda_1, \Phi, \overline{X}, \sigma, \Sigma, \delta_0, \delta_1)$ are given, the model is complete.

In order to identify the unobservable factors θ, it is assumed that $E(\theta) = 0$ and $Var(\theta) = \mathbb{I}$, a slight modification of the identification proposed by Dai and Singleton (2002), and that the latter have certain intertemporal causality ordering. That is, the factor of order i affects the subsequent one, but not the contrary. Following A&P, we also assume that $\sum_{M\theta} = 0$, which can be interpreted as the condition that the residues of the Taylor equation is unrelated to the macro variables. Then:

$$\Sigma = \begin{pmatrix} \Sigma_{MM} & 0 \\ 0 & I \end{pmatrix}, \;\; \overline{X} = \begin{pmatrix} \overline{X}_M \\ 0 \end{pmatrix}, \;\; \Phi = \begin{pmatrix} \Phi_{MM} & \Phi_{M\theta} \\ \Phi_{\theta M} & \Phi_{\theta\theta} \end{pmatrix}, \tag{5}$$

where $\Phi_{\theta\theta}$ is lower triangular.

2.1 Continuous Time Version

2.1.1 Pricing

We derive the pricing equations in the affine model. As usual, a probability space $(\Omega, \mathbb{F}, P)$ is fixed and no arbitrage is assumed. The price at time t of a zero coupon bond paying 1 at the maturity date $t + \tau$ is

$$P(t, \tau) = E^{\mathbb{Q}} \left[\exp \left(- \int_t^{t+\tau} r_t dt \right) \mid \mathbb{F}_t \right]. \tag{6}$$

The conditional expectation is taken under the equivalent martingale measure $\mathbb{Q}$.

The state of the economy is represent by a vector $X_t \in \mathbb{R}^p$. The short rate and risk premium process are given by $r_t = \delta_0 + \delta_1 \cdot X_t$ and $\lambda_t = \lambda_0 + \lambda_1 \cdot X_t$, where the state vector X_t follows a Gaussian process with mean reversion, which is a particular case of an affine dynamics and a continuous time equivalent of a Vector Autoregression (VAR). Under the objective P-measure,

$$dX_t = K(\xi - X_t)dt + \Sigma dw_t. \tag{7}$$

The $p \times p$ and $p \times 1$ parameters K and ξ represent the mean reversion coefficient and the long term mean short rate, and $\Sigma\Sigma^T$ is the instantaneous variance-covariance matrix of the p-dimensional standard Brownian shocks w_t.

Under the martingale measure $\mathbb{Q}$,

$$dX_t = K^\star(\xi^\star - X_t)dt + \Sigma dw_t^*, \tag{8}$$

where $dw_t^* = dw_t + \lambda_t dt$, w_t^* is a standard $\mathbb{Q}$-Brownian motion, and

$$K^\star = K + \Sigma\lambda_1,\ \xi^\star = K^{\star -1}(K\xi - \Sigma\lambda_0). \tag{9}$$

It can be shown (see Duffie, 2002) that bond price can be found using multifactor Feynman-Kac. If $P(t, \tau) = v(X_t, t, \tau)$, then $v(x, t, \tau)$ must satisfy

$$\mathbb{D}v(x, t, \tau) - r(x)v(x, t, \tau) = 0, \quad v(x, t, 0) = 1, \tag{10}$$

where $\mathbb{D}v(x, t, \tau) := v_t(x, t, \tau) + v_x(x, t, \tau) \cdot K^\star(\theta^\star - x) + \frac{1}{2} tr[\Sigma\Sigma^T v_{xx}(x, t, v)$, whose solution is $v(t, \tau, x) = e^{\alpha(\tau) + \beta(\tau) \cdot x}$, where

$$\beta\prime(\tau) = -\delta_1 - K^{\star\intercal}\beta(\tau), \tag{11}$$

$$\alpha\prime(\tau) = -\delta_0 + \xi^{\star\intercal}K^{\star\intercal}\beta(\tau) + \frac{1}{2}\beta(\tau)^\intercal\Sigma\Sigma^\intercal\beta(\tau). \tag{12}$$

Calculation of the explicit solution of the above system of ODE's is only possible in some special cases, such as when K is diagonal. However, Runge-Kuta numerical integration can solve equations (11) and (12) efficiently.

As a result, the yield is given by an affine function in the state variables, $Y(t,\tau) = -\frac{\alpha(\tau)}{\tau} - \frac{\beta(\tau)}{\tau} \cdot X_t$, or, defining $A(\tau) = -\frac{\alpha(\tau)}{\tau}$ and $B(\tau) = -\frac{\beta(\tau)}{\tau}$,

$$Y(t,\tau) = A(\tau) + B(\tau) \cdot X_t. \tag{13}$$

Stacking the equations for the K yield maturities, we arrive at a more concise expression:

$$Y_t = A + BX_t, \tag{14}$$

where $Y_t = (Y(t,\tau_1), ..., Y(t,\tau_K))^\intercal$.The factor loadings A and B will depend on the set of parameters $\Psi = (\delta_0, \delta_1, K, \xi, \lambda_0, \lambda_1, \Sigma)$, which are estimated according to the data being used.

2.1.2 Likelihood

The log-likelihood is the *log* of the density function of the sequence of observed yields $(Y_{t_1}, ..., Y_{t_n})$. To calculate it we must first find the transition density of $X_{t_i}|X_{t_{i-1}}$, integrating the equation (7):

$$X_{t_i|t_{i-1}} = (1 - e^{-K(t_i - t_{i-1})})X_{t_{i-1}} + e^{-K(t_i - t_{i-1})}\xi + \int_{t_{i-1}}^{t_i} e^{-K(t_i - u)}\Sigma dw_u. \tag{15}$$

The stochastic integral term above is Gaussian with mean zero and variance

$$E\left[\int_{t_{i-1}}^{t_i} e^{-K(t_i-u)}\Sigma dw_u\right]^2 = \int_{t_{i-1}}^{t_i} e^{-K(t_i-u)}\Sigma\Sigma^T (e^{-K(t_i-u)})^T du. \tag{16}$$

This means that $X_{t_i|t_{i-1}} \sim N(\mu_i, \sigma_i^2)$, where $\mu_i = (1 - e^{-K(t_i - t_{i-1})})X_{t_{i-1}} + e^{-K(t_i - t_{i-1})}\xi$ and σ_i^2 is the integral above. Since we use daily frequency, $dt = t_i - t_{i-1}$is very small, and thus the integral (16) can be well approximated using

$$\sigma_i^2 \simeq e^{-Kdt}\Sigma\Sigma^T (e^{-Kdt})^T dt. \tag{17}$$

In that case, we have $X_{t_i|t_{i-1}} = \mu_i + \sigma_i \, N(0, \mathbb{I})$,with $\sigma_i = e^{-Kdt}\Sigma\sqrt{dt}$.

Now suppose the vectors X_t and Y_t have the same dimension, that is, the number of yield maturities equals the number of state variables. Then, we can invert the linear equation (14) and find X_t as a function h of Y_t:

$$X_t = B^{-1}(Y_t - A) = h(Y_t). \tag{18}$$

Using change of variables, it follows that

$$\log f_Y(Y_{t_1}, ..., Y_{t_n}; \Psi) = \log f_X(X_{t_1}, ..., X_{t_n}); \Psi) + \log |\det \nabla h|^n \tag{19}$$

$$= \sum_{i=2}^{n} (\log f_{X_{t_i}|X_{t_{i-1}}}(X_{t_i}; \Psi) + \log |\det \nabla h|). \tag{20}$$

However, this procedure clearly restricts the number of yield maturities that can be used. If we wanted to use more available data, the additional yields would make the model singular. One solution is to follow Chen and Scott (1993), and add measurement errors to some yields. We choose this method in our continuous time versions. We select p maturities out of n to be priced without error. Let Y_t^1 represent the set of those yields at a given time. The other yields are denoted by Y_t^2, and they will have independent normal measurement errors $u(t,\tau) \sim N(0,\sigma^2(\tau))$.

To incorporate macro factors, we extend our state vector of our economy to include observable macro variables M_t, that is, $X_t = (M_t, \theta_t)$. Following A&P, the short rate will be a combination of a Taylor Rule and the affine model, $r_t = \delta_0 + \delta_{11} \cdot M_t + \delta_{12} \cdot \theta_t$. Then, similar pricing equations lead to

$$Y(t,\tau) = A(\tau) + B^M(\tau)M_t + B^\theta(\tau)\theta_t. \tag{21}$$

It turns out that

$$\begin{bmatrix} M_t \\ Y_t^1 \\ Y_t^2 \end{bmatrix} = \begin{bmatrix} 0 \\ A^1 \\ A^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ B^{M\,1} & B^{\theta\,1} & 0 \\ B^{M2} & B^{\theta\,2} & 1 \end{bmatrix} \begin{bmatrix} M_t \\ \theta_t \\ u_t \end{bmatrix}. \tag{22}$$

Denote by h the function from the state (X_t, u_t) vector to (M_t, Y_t^1, Y_t^2). One obtains θ_t inverting on Y_t^1:

$$\theta_t = (B^{\theta\,1})^{-1}(Y_t^1 - A^1 - B^{M\,1}X_t^M). \tag{23}$$

Then the log likelihood $\mathcal{L}$ is

$$\mathcal{L} = -(n-1)\log|\det B^{u\,1}| + \sum_{t=2}^{n} \log f_{X_t|X_{t-1}}(X_t;\Psi) + \log f_u(u_t), \tag{24}$$

as before.

2.2 Discrete Time Version

2.2.1 Pricing

Following the A&P approach, we derive the discrete time equation. The price at time t of an asset V_t that pays no dividend is

$$V_t = E^{\mathbb{Q}}[\exp(-r_t)V_{t+1}|\mathbb{F}_t]. \tag{25}$$

Again we work under no arbitrage condition, with $\mathbb{Q} \sim \mathbb{P}$ being the martingale measure and $\mathbb{F}_t$ the filtration. The short rate and the risk premium will be again affine functions of the state vector $X_t \in \mathbb{R}^p$, that is, $r_t = \delta_0 + \delta_1 X_t$ and $\lambda_t = \lambda_0 + \lambda_1 X_t$, where the dynamics of the state vector is a multifactor vector autoregression

$$X_t = \mu + \Phi X_{t-1} + \Sigma\epsilon_t. \tag{26}$$

Denote by ξ_t the Radon-Nikodym derivative $\frac{dQ}{dP} = \xi_t$. A discrete time "version" of Girsanov theorem is assumed setting $\xi_{t+1} = \xi_t \exp(-\frac{1}{2}\lambda_t \cdot \lambda_t - \lambda_t \epsilon_{t+1})$, where $\{\epsilon_t\}$ are independent normal errors. Then, the Pricing Kernel will be $m_t = \exp(-r_t)\frac{\xi_{t+1}}{\xi_t}$, so that the price of a zero coupon bond maturing n+1 periods ahead is $p_t^{n+1} = E[m_{t+1}p_{t+1}^n]$. It can be proved by induction that the price of bond will be exponential affine:

$$p_t^n = \exp(\alpha_n + \beta_n X_t), \tag{27}$$

where:

$$\begin{gathered}\alpha_1 = -\delta_0,\ \beta_1 = -\delta_1,\\ \alpha_{n+1} = -\delta_0 + \alpha_n + (\mu^\intercal - \lambda_0^\intercal\Sigma)\beta_n + \tfrac{1}{2}\beta_n^\intercal\Sigma^\intercal\Sigma\beta_n,\\ \beta_{n+1} = -\delta_1 + (\Phi - \lambda_1^\intercal\Sigma)\beta_n.\end{gathered} \tag{28}$$

Then $Y_t^n = -\log p_t^n/n = A_n + B_nX_t$, where $A_n = -\alpha_n/n$ and $B_n = -\beta_n/n$.Forming a vector of yields, we arrive at the same expression as in the continuous case,

$$Y_t = A + BX_t. \tag{29}$$

2.2.2 Model Specification

The yield curve is described by $X = (M, \theta)$. The observation equation relates the evolution of the yield curve to the state $Y|M,\theta$ through matrices A and B, whose coefficients depend on the monetary rule that determines the short rate given the state of the economy, the affine risk premium, and the idiosyncratic variance error σ:

$$\begin{gathered}Y_t = A(\delta_0, \Sigma\Sigma^T, \mu^*, \Phi^*) + B(\delta_1, \Phi^*)X_t + u_t,\ u_t \sim (0, I_\sigma)\\ (M_t, \theta_t) = X_t = \mu + \phi X_{t-21} + \Sigma\epsilon_t,\ \epsilon_t \sim N(0, I)\\ r_t = Y_t^1 = \delta_0 + \delta_1 X_t + u_t^1,\end{gathered} \tag{30}$$

where now measurement errors u_t are added to all maturities, and $\mu^* = \mu - \Sigma^T\lambda_0$, $\Phi^* = \Phi - \lambda_1^T\Sigma$. The parameters (μ, Φ) characterize the $\mathbb{P}$-dynamics of the state variables, (δ_0, δ_1) the monetary rule that determines the short rate given the state of the economy, (λ_0, λ_1) the risk premia describing the dynamics of the cross-section, and $\Sigma\Sigma^\intercal$ the covariance among the shocks that determine the state of the economy. Following Johannes and Polson (2003), (μ^*, Φ^*) are directly estimated, from which the premia is inferred.

The no arbitrage equations depend on Φ^m, where m is the yield maturity. In the case of daily data, m can be a very high number, such as $m = 21*36$ for the 3 year maturity, which may imply uncontrolled approximation errors. Our strategy to avoid it was to use a monthly model, in which the transition occurs given a 1 month (21 commercial days) lagged variable, with 21 replications. In this way, we have a monthly model with the same volume of information of the daily data.

In order to identify monetary factors, we imposed condition (5) in each iteration, implying that only a subset of the elements of Σ is free. Let Σ^* be this subset. The parameters are $\psi = (\mu, \phi, \sigma, \theta, \zeta)$, where $\zeta = (\delta_0, \delta_1, \mu^*, \Phi^*, \Sigma^*\Sigma^{*\intercal})$, and the likelihood is

$$p(Y|M,\psi) = p(Y|\theta, M, \psi)p(\theta|M,\psi)p(\psi), \tag{31}$$

whose components are described by:

$$
\begin{aligned}
&p(Y|\theta, M, \psi) = \prod_t p(Y_t|\theta_t, M_t, \psi) = -\frac{1}{2}\left[N\sum_i \ln(\sigma_i^2) + \sum_t \sum_i (\frac{u_{it}^2}{\sigma_i^2})\right],\\
&u_{it} = Y_{it} - A_i(\delta_0, \Sigma^*\Sigma^{*\intercal}, \mu^*, \Phi^*) - B_i(\delta_1, \Phi^*)X_t,\\
&P(\theta|\psi) = \prod_t P(\theta_t|\theta_{t-1}, M_{t-1}, \psi) = -\frac{1}{2}\left[N\sum_i \ln|\Sigma\Sigma^\intercal| + \sum_t^\intercal \epsilon_t^\intercal(\Sigma^*\Sigma^{*\intercal})^{-1}\epsilon_t\right],
\end{aligned}
\tag{32}
$$

where $\sum \epsilon_t = X_t - \mu + \Phi X_t$. A non-informative prior $p(\psi)$ is assumed. In case there are unobservable variables, we could eliminate the monetary factors by integration[1].

2.2.3 Other Discrete Versions

Alternative models can be specified relaxing the no arbitrage condition. We present two alternatives for comparison purposes: 1) A model without restrictions over the matrices (A, B), which correspond to multivariate factor model of the time series literature (West and Harison, 1997), 2) A model following Diebold and Li (2006).

- Factor Model:

$$
\begin{aligned}
Y_t &= A + BX_t + u_t,\ u_t \sim N(0, \mathbb{I}_\sigma)\\
X_t &= (M_t, \theta_t)^\intercal = \mu + \Phi X_{t-21} + u_t,\ u_t \sim N(0, \Sigma\Sigma^\intercal)
\end{aligned}
\tag{33}
$$

- Modified Diebold-Li Model:

  Has unobservable state variables $\{\theta_{1t}, \theta_{2t}, \theta_{3t}\}$,

$$
\begin{aligned}
Y_t^n &= \theta_{1t} + \left(\frac{1-e^{-\lambda n}}{\lambda n}\right)\theta_{2t} + \left(\frac{1-e^{-\lambda n}}{\lambda n} - e^{-\lambda n}\right)\theta_{3t} + u_{nt}\\
&= \theta_{1t} + B_{2n}(\lambda)\theta_{2t} + B_{3n}(\lambda)\theta_{3t} + u_{nt},\\
\text{where}\ \ \theta_t &= \mu + \Phi\theta_{t-21} + \epsilon_t,\ \epsilon_t \sim N(0, \Sigma\Sigma^\intercal)
\end{aligned}
\tag{34}
$$

We propose a modified Diebold-Li model incorporating macro variables into the model in the same way as Ang and Piazzesi. Macro factors coefficients are given by Nelson-Siegel model and Φ is assumed upper triangular in order to make the model identifiable:

$$
Y_t = B(\lambda)X_t + \epsilon_t,\ \epsilon_t \sim N(0, \mathbb{I}_\sigma). \tag{35}
$$

[1] $p(Y|M,\psi) = \int p(Y|\theta, M, \psi)p(\theta|M,\psi)p(\psi)d\theta = -\frac{1}{2}\left[\sum_i \log(|Q_t|) + \sum_t \sum_i (u_t^\intercal Q_t^{-1} u_t)\right]$, where $u_t = Y_{it} - A_i - B_i(M_t, E(\theta_t|t-1))$.

3 Inference

3.1 Continuous Time

The parameters are found maximizing the log-likelihood with respect to the parameters given the series of yields. The maximum likelihood estimation produces asymptotically consistent, non-biased and normally distributed estimators. Let $\mathcal{L} = \log f_Y$.When $T \to \infty, \hat{\psi} \to \psi$ a.s. and $T^{\frac{1}{2}}(\hat{\psi}-\psi) \to N(0,\Omega)$ in distribution. An estimator for Ω^{-1} is the empirical Hessian

$$\hat{\Omega}^{-1} := -\frac{1}{n}\sum_{t=1}^{n}\left(\frac{\partial^2 \mathcal{L}_t(Y;\hat{\psi})}{\partial\psi^2}\right),$$

where $\mathcal{L}_t$ represents the likelihood of the vector with t elements. More details can be found in Davidson and Mackinnon (1993, Chapter 8).

We calculate the confidence intervals for the parameter estimations using equation the empirical Hessian and the Central Limit Theorem. If the number of observations n is large enough, then the variance of $\hat{\psi} - \psi$ will be given by the diagonal of $N(0, \Omega/n)$.

3.2 Discrete Time

The distribution of the parameters,

$$p(\theta|Y, M_t, \psi) \propto p(Y|\theta, M_t, \psi)p(\theta|M_t, \psi)p(\psi), \tag{36}$$

cannot be derived analytically, but the Clifford-Hammersley theorem guarantees that the recursive sampling of subsets of parameters, obtained from the complete conditional distributions, converges to the joint distribution. The subsets are chosen in a convenient way such that the subproblems have, when possible, analytical solutions and known complete conditional distributions, as is the case of subproblems 1-3 bellow. These problems correspond to, respectively, an estimation of a VAR model, the variance of known random variables, and the extraction of unobservable factors from a multivariate dynamic model. The subproblem (4) relative to ζ does not have known expression and its distribution will be derived utilizing the Metropolis-Hastings rejection method (Gamerman, 2001, and Johannes and Polson, 2003), with a proposal obtained from a normal distribution, centered on the value of the previous iteration, and with an arbitrarily fixed variance such that the acceptance rate is in the interval [0.3, 0.8]. The distributions calculated in each step of the algorithm are:

1. $(\mu^w, \Phi^w) \sim p(\mu, \Phi|\sigma^w, \zeta^w, \theta^w)$,
2. $\sigma^w \sim p(\sigma|\mu^w, \Phi^w, \zeta^w, \theta^w)$,
3. $\theta^w \sim p(\theta|\mu^w, \Phi^w, \zeta^w, \sigma^w)$,

4. $\zeta_i^w \sim p(\zeta_i|\zeta_{-i}^w, \mu, \Phi, \Sigma, \sigma, \theta)$,

We have:

Subproblem1: $p(\mu, \Phi|\sigma^w, \zeta^w, \theta^w) \sim N((X^\intercal X)^{-1}X^\intercal X^*, (X^\intercal X)^{-1} \otimes \Sigma)$, where $X = (X_1, ..., X_{T-1})^\intercal$, $X^* = (X_2, ..., X_T)^\intercal$, $X = (M, \theta)$.

Subproblem2: $p(\sigma|\mu, \Phi, \zeta, \theta) \sim \mathcal{IG}(diag(\epsilon^\intercal \epsilon))$, where $\epsilon = Y - A - BX$, and $\mathcal{IG}$ is the inverted gamma distribution.

Subproblem3: $p(\theta|\mu, \Phi, \sigma, \zeta) = \prod_t p(\theta_t|\mu, \Phi, \sigma, \zeta)$, where $p(\theta_t|\mu, \Phi, \sigma, \zeta) = p(\theta_t|D_T) \sim N(h_t, H_t)$ is the FFBS algorithm defined in the Appendix.

Subproblem4: $p(\zeta_i|\zeta_{-i}, \mu, \Phi, \sigma, \theta) = p(Y|\theta, M, \psi)p(\theta|M, \psi)p(\psi)$.The proposal used in the Metropolis algorithm is: $p(\zeta_i|\zeta_{i-1}, \mu, \Phi, \sigma, \theta) \sim N(\xi_i^k, c)$ and accepts if $p(Y|\xi_i^k) - p(\theta|\xi_i^{k-1}) > u$, $u \sim U(0, 1)$.

3.2.1 Unrestricted Model

In this model all the elements of (A, B) are estimated. No arbitrage conditions are not used. Step 4 is substituted by:

$$P(A, B|\mu, \Phi, \sigma, \theta) = N((X^\intercal X)^{-1}X^\intercal Y, (X^\intercal X)^{-1} \otimes \sigma^2). \tag{37}$$

3.2.2 Modified Diebold-Li

The step 4 is substituted by the following two steps:

4-1. $p(\lambda|\mu, \Phi, \sigma, \theta)$ conditional distribution approximated by the Metropolis method with proposals obtained from symmetric distributions centered on the parameter of the previous iteration:

Proposal: $p(\lambda|\mu, \Phi, \sigma, \theta) \sim N(\lambda^k, c)$, accepts if $p(Y|\lambda^k) - p(Y|\lambda^{k-1}) > u$, $u \sim U(0, 1)$.

4-2. The observation equation can be written as $Y_t - B_\theta \theta_t = \widetilde{Y}_t = B_M M_t + e_t$. Then

$$P(B_M|\mu, \Phi, \sigma, \theta, \lambda) = N((M^\intercal M)^{-1}M^\intercal \widetilde{Y}, (M^\intercal M)^{-1} \otimes \sigma^2). \tag{38}$$

4 Results

Two contracts market by the BM&F were used to measure the expected inflation and the yield curve. The first one is the `DIxPRE` swap, whose prices for various maturities provides a measure of the term structure, that was defined with the maturities $\{1, 2, 3, 6, 9, 12, 18, 24, 36\}$-months. The second asset - `INPCxDI` Swap - provide the difference between the rate of inflation measured by the consumer price index and the floating interest rate observed at the contracted maturity. The ratio between the earnings of the latter asset and of the corresponding DIxPre swap was considered a measure of the expected inflation for that maturity. However, this ratio contains a risk premium that was supposed constant and thus disregarded. Considering the volatility of the series, a maturity of 6 months for the expected inflation was chosen. For reasons unknown to us, the expectations of lower maturities presented high volatility.

Following Litterman and Scheinkman (1991), we analyzed the yield curve in Brazil, which indicated that 99% of the variance of the 9 yield maturities in our sample can be described by two principal components (90% and 9% for the first and second component). This motivated a model with 2 unobserved monetary factors. Then, the main sources of nominal shocks in the economy, the log of the nominal exchange rate and the expected inflation rate, were added. The independent structural shocks associated to those variables were identified supposing that the innovation of the exchange rate determines the innovation of the expected inflation but not the other way around.

With those hypothesis, the model has 4 exogenous independent shocks: the exchange rate shock and the expected inflation shock, which affect the short rate, and other two shocks corresponding to the innovations of the two unobservable monetary factors.

Since this model incorporates a "Taylor" rule, the effect of an unexpected rise in inflation is linked to the reaction of the monetary authority with respect to the policy of inflation control. On the other hand, the identified monetary factors have different characteristics. The unobserved factor 1 is highly correlated to the difference between the long and the short rate, which we call *slope*. The unobserved factor 2 is highly correlated to the mean value of the rates, which we denote by *level*. This is shown in the tables that summarize the results. Although our model differs from the model proposed by Rudebusch and Wu (2004), RW, the estimated monetary factors show similar characteristics, and based on this argument we will interpret the factors and the corresponding shocks in the same way as RW. Their model contains two unobserved monetary factors - slope and level -, a MA reaction rule - Taylor rule -, and a transition equation specified with restrictions among the parameters, derived from a “macro structural” model.

RW interpreted the innovation that persistently increases all the yield rates as the shock over the preferences of the MA with respect to the level of inflation, that is, as an alteration of the Inflation Target, even though an implicit one, for the FED does not have a explicit inflation target up to th present date. The innovation of the factor 2 was interpreted as an alteration of the determinants of the monetary policy, which, in the case of the USA, could be caused by credit crunchs, price misalignments or increases of risk perception. In other words, it is an innovation that is not linked to a movement of the inflation. RW associate this type of shock to FED’s reaction to the recurring financial market crises, such as the 1997 bubble or the consequences of the terrorist attack of September 2001. We shall keep the interpretations in the case of Brazil. A domestic example was the crisis on the eve of the presidential election in 2002, when there existed a rejection by the local market of the government bonds maturing after the election.

Due to estimability issues and to the possibility of overparameterization, A&P impose restrictions over the transition matrix Φ, the premium λ and the correlation matrix Σ. We follow their option to set $\Sigma_{\theta M} = 0$ (uncorrelated contemporaneous shocks), but we choose to experiment with other options regarding Φ and λ. We test a version in which $\Phi_{\theta M} = \lambda_{\theta M} = 0$, making the monetary factors become independent of the macroeconomic conditions. The macro shocks affect the monetary factors exclusively through the Taylor rule. We call this specification Unilateral, as opposed to the also studied unrestricted Bilateral specification. It is an empirical question to evaluate the effects of those restrictions over the results of the model.

Finally, a few comments about how we deal with local maxima in the estimation process. In the optimzations, we start from the maximum of more restricted models. In the MCMC, we constructed 10 chains and choose that one that presents the highest mean value of the loglikelihood after the chain converges.

4.1 Evaluating A&P Model's specifications

To analyze the performance of the model and the quantity of information brought by each macro variable, the model was estimated using 4 alternatives: (m) one purely monetary, $X = \theta$; (i) one including expected inflation, $X = (i, \theta)$; (e) one including exchange rate, $X = (e, \theta)$; and (ei) one including both, $X = (e, i, \theta)$. These 4 models were estimated in the unilateral and bilateral versions, in both the continuous (C) and discrete time (D) versions.

The models will be compared considering: 1) log likelihood; 2) standard deviation, measured out-of-sample, of a 1-month forecasting error of selected maturities, normalized by the standard deviation of a model that follows a random walk, also know as Theil-U. That is,

$$\text{Theil-U} = \left(\frac{\sum_t (Y_t - \widehat{Y}_{t|t-21})^2}{\sum_t (Y_t - Y_{t-21})^2} \right)^{\frac{1}{2}}; \tag{39}$$

3) The mean value M(In) of all maturities of the measure just defined, but in–sample, 4) Correlation among the factors and the level and slope of the yield curve; and 5)Mean standard deviation of measurement error M(σ). We remark that given the number of observations (770), the ordering of performance according to likelihood will not be changed if Akaike or Baysean Informatin Criterion are use instead. Table 1 presents the results.

Table 1a: Comparison of specifications

Discrete	m	i-U	i-B	e-U	e-B	ei-U	ei-B
LogLik$*10^{-4}$	3.34	4.15	4.04	4.10	3.31	4.28	4.01
N-param	19	28	32	28	32	42	50
M(In)	1.20	0.97	0.96	0.97	1.06	0.92	0.91
Theil–U(1m)	4.53	0.91	2.34	0.87	1.27	1.16	1.83
Theil–U(6m)	4.69	1.01	1.06	1.09	0.98	1.96	1.24
Theil–U(12m)	3.37	1.02	1.05	1.01	0.83	0.87	1.01
Theil–U(36m)	2.40	1.22	2.06	1.29	1.25	1.86	1.43
C(θ_1,Slope)	0.92	0.97	0.87	0.98	0.90	0.99	0.94
C(θ_2,Level)	0.96	0.74	1.00	0.64	0.98	0.94	0.86
M(σ) basis pts	57	63	42	65	59	128	56

Table1b: Comparison of specifications

Continuous	m	i-U	i-B	e-U	e-B	ei-U	ei-B
LogLik$*10^{-4}$	3.48	3.88	3.90	3.80	3.81	4.20	4.24
N-param	17	26	30	26	30	40	48
M(In)	0.96	0.95	0.94	0.97	0.95	0.94	0.92
Theil–U(1m)	2.39	3.18	3.13	2.76	3.59	1.58	5.32
Theil–U(6m)	1.44	2.15	1.55	1.78	0.83	0.62	1.24
Theil–U(12m)	1.05	0.82	1.34	0.84	2.74	1.54	2.00
Theil–U(36m)	3.78	3.33	4.71	3.71	6.14	4.97	5.92
C(θ_1,Slope)	0.90	0.92	0.83	0.93	0.70	0.85	0.55
C(θ_2,Level)	0.68	0.90	0.88	0.85	0.86	0.63	0.89
$M(\sigma)$ basis pts	45	42	42	44	43	37	40

The results show that:

1. The in-sample adherence and the out-of-sample forecasting performance of both versions are similar.
2. In both versions, the monetary factor θ_1 is highly correlated to the slope and the monetary factor θ_2 to the level of the yield curve.
3. The likelihood indicates that:
   (a) The macro variables add information.
   (b) Inflation adds more information than the exchange rate.
   (c) The two macro variables combined add more information than each one alone.
   (d) In the discrete model with 2 macro factors, the bilateral model (B) is worse than the unilateral model (U).
   (e) The continuous bilateral model is marginally better than the unilateral.
4. The models do showed low forecasting performance for the majority of the maturities, in both versions and in the various specifications.

The log likelihood in the continuous time versions were calculated at the point of maximum, and in the discrete versions as the mean value. The restriction that characterizes the unilateral models do not impose significative reductions in terms of the information criterion in the continuous case. However, in the discrete case, the restrictions may even improve the expected log likelihood. This suggests the difficulty of estimating the parameters of the bilateral model.

The trajectory of the state variables and of the yield curve is explained by the identified shocks assuming the following exogeneity ordering: exchange rate shock, inflation shock and unobservable factor 1 (slope) shock. The following table shows the proportion of the variance of each variable that is explained by each shock, 18 months ahead.

Table 2a: Variance Decomposition 18 months ahead

	exchange rate shock				inflation shock			
	U/D	B/D	U/C	B/C	U/D	B/D	U/C	B/C
e	0.01	0.31	0.05	0.77	0.01	0.57	0.05	0.04
i	0.01	0.03	0	0.21	0.01	0.49	0.40	0.64
1m	0	0.03	0	0.22	0	0.66	0	0.24
9m	0	0.04	0	0.21	0	0.61	0	0.07
36m	0	0.04	0	0.28	0	0.58	0	0.01
	slope shock				level shock			
	U/D	B/D	U/C	B/C	U/D	B/D	U/C	B/C
e	0.89	0.11	0.55	0.19	0.09	0.26	0.10	0
i	0.06	0.07	0.53	0.14	0.93	0.36	0.05	0
1m	0.66	0.13	0.78	0.20	0.34	0.18	0.22	0.34
9m	0.77	0.13	0.96	0.40	0.23	0.23	0.04	0.33
36m	0.85	0.12	1	0.47	0.15	0.26	0	0.24

The results show that:

1. Most of the path of the macro variables and of the yields of the maturities is explained by the monetary factors, implying that inflation shock explains a negligible fraction of the path. This indicates that the Taylor rule is not an important mechanism.

2. The exception is the bilateral discrete model, where the inflation explains most of the variance, and indicates the importance of the Taylor rule

3. In the case of the bilateral continuous model, the inflation explains 64% of the inflation path, suggesting the importance of the effect of the Taylor rule, and the exchange rate explains 77% of the trajectory of the of the exchange rate. This is an indication that although the exchange rate affects the other variables of the model, it is mostly not affected by them.

4.2 Identifying Short Run Market Reaction Function

The short run equation represents the reaction of the market given a change in the state of the economy: $r_t = \delta_0 + \delta_M M_t + \delta_\theta \theta_t$. Ang, Dong and Piazzesi (2005) show that different restrictions over δ accommodates the identification of various types of Taylor rules. For example, they test the following 3 rules: (S) Standard: if $\delta_\theta = 0$, the monetary authority decides based on the present value of the variables; (F) Forward-looking: if $\delta_M = 0$, it decides based on the (infinite with no discount) future expectation of the macro variables, and (B) Backward-looking: if there are no restrictions, it decides smoothing the monetary policy.

In our model, the same equation cannot be interpreted as a monetary authority reaction function, but it is a key element in the pricing and works equivalently. This fact motivated us to evaluate the same restrictions on it. Up to now, only the unrestricted form was used. Using the model with the two macro variables (ei), Table 3 presents the effect of imposing restrictions on the short run equation for the unilateral and bilateral specifications. The unilateral forward model was taken away because in this case the macro and the unobservable factors are completly independent.

The restriction, which only take into account the present value of the macro variables, present the worse performance in both versions in terms of likelihood. The irrestricted specification presents the best performance in both versions and all identifications.

During the sample period, the Brazilian MA implemented a regime of explicit inflation target, which was considered successful in the opinion of most analysts. Thus, we would expect that the inflation shock explains an important fraction of the trajectory of the inflation and of the short rate. The other rates, being nominal variables, should also be affected in some degree by this shock. It is worth mentioning that during the period the inflation target remained relatively stable, which implies less importance of level shocks.

In view of the above comments, we conclude that if the market expectation of inflation is a good proxy for the MA expected inflation, inflation shock should explain most of long run variance. However, only the discrete bilateral models with irrestricted specification worked in that way, and the results of the other models are incorrect due to identification, estimation or restriction problems.

On the other hand, if the our proxy for the MA inflation is incorrect, then the other models are correct and most variance should be explained by monetary factor shocks, specially slope shock.We remark that the restricted version of short run equation presents worser perfomance than the irrestricted, and that althought the inflation effect on the short rate is almost null, it is larger on the longer maturities.

Table3: Comparison of specifications

Discrete	B-U	B-B	S-B	F-B	B-U	B-B	S-B	F-B
LogLik$*10^{-4}$	4.28	4.01	3.25	3.78	4.20	4.24	4.20	4.24
N-param	42	50	48	48	40	48	46	46
M(In)	0.92	0.91	1.35	1.41	0.97	0.93	1.00	0.93
Theil–U(1m)	1.16	1.83	3.68	9.79	1.58	5.32	0.58	5.77
Theil–U(6m)	1.96	1.24	3.97	5.68	0.62	1.24	0.71	1.43
Theil–U(12m)	0.87	1.01	3.33	2.85	1.54	2.00	1.76	1.91
Theil–U(36m)	1.86	1.43	4.02	1.90	4.97	5.92	4.92	5.90
C(θ_1,Slope)	0.99	0.94	0.90	0.87	0.85	0.55	0.87	0.50
C(θ_2,Level)	0.94	0.86	0.86	0.98	0.63	0.89	0.94	0.91
$M(\sigma)$ basis pts	150	53	175	56	37	40	42	40

Table 4a: Variance Decomposition 18 months ahead. Macro shocks

	exchange rate shock							
	discrete				continuous			
	B-U	B-B	S-B	F-B	B-U	B-B	S-B	F-B
e	0.01	0.05	0.08	0	0.31	0.77	0.16	0.78
i	0.01	0.08	0.08	0.01	0.03	0.21	0.02	0.22
1m	0	0.03	0.08	0	0	0.22	0	0.21
9m	0	0.04	0.08	0	0	0.21	0	0.2
36m	0	0.04	0.07	0	0	0.28	0	0.27

	inflation shock							
	discrete				continuous			
	B-U	B-B	S-B	F-B	B-U	B-B	S-B	F-B
e	0.01	0.57	0.66	0.07	0.05	0.04	0.03	0.03
i	0.01	0.49	0.7	0.25	0.4	0.64	0.26	0.64
1m	0	0.66	0.72	0.05	0	0.24	0	0.21
9m	0	0.61	0.62	0.08	0	0.07	0	0.07
36m	0	0.58	0.69	0.09	0	0.01	0	0.02

Table 4b: Variance Decomposition 18 months ahead. Factor Shocks.

	slope shock							
	discrete				continuous			
	B-U	B-B	S-B	F-B	B-U	B-B	S-B	F-B
e	0.89	0.11	0.09	0.92	0.55	0.19	0.59	0.19
i	0.06	0.07	0.1	0.7	0.53	0.14	0.56	0.14
1m	0.66	0.13	0.08	0.94	0.78	0.2	0.03	0.21
9m	0.77	0.13	0.08	0.91	0.96	0.4	0.65	0.38
36m	0.85	0.12	0.09	0.9	1	0.47	0.8	0.45

	level shock							
	discrete				continuous			
	B-U	B-B	S-B	F-B	B-U	B-B	S-B	F-B
e	0.09	0.26	0.17	0.01	0.1	0	0.22	0
i	0.93	0.36	0.12	0.04	0.05	0	0.17	0
1m	0.34	0.18	0.13	0	0.22	0.34	0.97	0.36
9m	0.23	0.23	0.21	0.01	0.04	0.33	0.35	0.35
36m	0.15	0.26	0.15	0.01	0	0.24	0.2	0.26

4.2.1 Evaluating the No Arbitrage restrictions

The observation equation of the A&P model relates the state of the economy with the term structure via an affine model where the coefficients are restricted by the no arbitrage conditions and an affine risk premium. Those hypothesis can be violated by various reasons. For example, the market may not be sufficiently ample and liquid to guarantee no arbitrage, or the premium may not vary as an affine function of the state variables, or the volatility should be considered as a state variable. The assumption of no arbitrage can be particularly strong for a market like the Brazilian, where the operations have been, most of the time, concentrated over the shorter rates. On the other hand, while the affine specification for the premium is used because its convenience, it has not yet been justified theoretically in terms of underlying preferences.

The assumptions were tested in an indirect way, for the Brazilian financial market, estimating discrete-time models with the observation equation defined in the unrestricted way (I) and in the modified Diebold-Li specification (DL). Both unilateral and bilateral specifications were considered.

In the present case, an additional comparison criterion was used. Gelfand and Gosh (1998), GG, proposed a criterion, proper for Monte Carlo inference methods, which allow the comparison of state variable models with many parameters, and consists of a weighted sum of the variance of the adjustment error and of the forecasting variance. In our case, since the errors of the yield curve are independent, we simply summed those variances for all maturities:

$$\mathrm{GG} = \sum_i \sum_t \left(Y_t^i - E(Y_t^i|\Omega)\right)^2 + \sum_i \sum_t \frac{1}{N_w} \sum_w \left(E(Y_t^i|\psi^w) - E(Y_t^i|\Omega)\right)^2, \tag{40}$$

$$E(Y_t^i|\Omega) = \frac{1}{N_w} \sum_w E(Y_t^i|\psi^w), \tag{41}$$

where w are realizations of the random variables.

Table6: Comparison of specifications

	Unilateral			Bilateral		
	AP	I	DL	ap	i	DL
LogLik$*10^{-4}$	4.28	5.92	3.51	4.01	4.93	3.65
N-param	42	72	28	50	76	32
M(In)	0.92	0.93	1.08	0.91	0.90	1.12
Theil–U(1m)	1.16	1.28	11.20	1.83	1.42	11.21
Theil–U(6m)	1.96	0.69	1.45	1.24	1.00	2.02
Theil–U(12m)	0.87	0.64	1.36	1.01	0.68	1.59
Theil–U(36m)	1.86	1.58	2.22	1.43	1.59	1.73
C(θ_1,Slope)	0.99	0.63	−0.76	0.94	0.97	−0.88
C(θ_2,Level)	0.94	0.99	0.98	0.86	1.00	0.96
GG$*10^{-3}$	1.3	1.1	1.2	1.4	1.1	1.2
$M(\sigma)$ basis pts	38	27	11	56	28	101

The unrestricted model presents the best performance according all to criteria, and the second best is A&P according to likelihood or DL according to GG criterion. DL model has only one parameter for pricing while A&P has a lot depending on the number of state variables. Even so, both models show similar performance, with DL presenting better performance according G&G criterion, because it penalizes parameter uncertainty. This may indicate that A&P model needs additional restrictions on premium parameters, but we have no economic criteria to define them. Bilateral models have more premium parameters and may show worse forecasting performance than unilateral models, but identifies inflation shocks according to expected. We are confronted by a dilemma between comprehensiveness and econometric estimability.

4.3 Impulse Response Function and Factor Loading

In order to analyze the dynamic properties of the model and effects of schocks along the maturities we selected one representative specification from the discrete and the continuous time versions. The model with the best performance in the continuous case is the bilateral, with unrestricted short run and two macro variables. In the discrete case the same specification was chosen. This version did not present the best performance, but correctly identified the inflation effects. The IRF results are shonw in Figure 1 and 2. The graphs on the left corresponds to the discrete version, and on the right the continuous. In each graph, it contains the response of all the variables to the indicated shock. Our analysis indicated that:

1. The exchange rate shock does not affect much other variables
2. The inflation shock causes a raise of the all the rates, affecting the slope, and a reduction of the exchange rate (dollar devaluation).
3. The slope shock have effects similar to the inflation shock, since it alters the slope of the yield curve and reduces the exchange rate. This resemblance suggests a difficulty of distinguishing the two shocks.

4. The level shock, as expected, rises all the nominal variables, and is associated to the loosening of the inflation target.

Factor loading - the matrix B - represents the effect of state variables over the maturities and is presented in Figure 5 for the continuous and discrete cases.

1. The level factor loading, as expected, is flat;

2. The slope factor loading decays along the maturities;

3. The macro schocks affect more the longer maturities than the shorter ones, indicating that the market antecipates the MA's reaction.

Summarizing, the response function of the variables to the identified shocks presented results according to expected. Both versions showed qualitatively similar results, although quantitatively different as shown by the variance decompositions.

5 Conclusion

This exercise estimated, using Brazilian financial market data, a no arbitrage term structure model proposed by A&P in discrete and continuous versions, with different specifications in order to identify the legitimacy of the restrictions over the dynamics and over the Taylor rule. The adherence and the forecasting performance of the model was evaluated comparing it to the corresponding random walk model. Furthermore, two factor models not complying with no arbitrage conditions were estimated. We learned among other things that:

1. The continuous and the discrete versions turned out to show qualitatively similar results, even though they relied on different equations and estimated using different inference methods, which suggests the robustness of the results.

2. The irrestricted short run equation, which contains the effect of the monetary factors, showed the best performance, suggesting that the market do not take into account only the current value of the macro factors.

3. The restrictions over the dynamics of state variables which we call unilateral implies an inadequate identification of inflation shocks.

4. The forecasting performance of the A&P model was low. It was worse than the unrestricted model in spite of having less parameters, under the out-of-sample forecasting and G&G criteria. This result casts doubts about the effectiveness of the no arbitrage condition and of affine premium for Brazilian data.

5. The estimated A&P model presented a strong link between macro and financial variables and identified structural shocks that works in a reasonable way, being able to evaluate the effect of monetary shocks over the yield curve.

Our exercise leaves some open methodological questions, such as how to restrict premium parameters to get better forecasting performance and to avoid identification problems, why unrestricted models showed better forecasting perfomance than no arbitrage models, whether adding stochastic volatility as state variable would bring better forecasting and economic perfomance. Future versions will address other questions, such as investigating if international liquidity and domestic fundamentals affect the term structure of credit spreads, measured by the yields of the external Brazilian bonds.

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A Appendix

A.1 Kalman Filter and FFBS algorithm

Here we present the Kalman Filter and the FFBS algoritm of the Dynamic Linear Model (DLM). Mixed because part of the components are observed (M). Defining

$$\begin{array}{l} Y_t = A + B_M M_t + B_\theta \theta_t + e_t,\ e_t \sim N(0, \mathbb{I}_\sigma), \\ M_t = \mu_M + \phi_{MM} M_{t-1} + \phi_{M\theta}\theta_{t-1} + u_t^M, \\ \theta_t = \mu_\theta + \phi_{\theta M} M_{t-1} + \phi_{\theta\theta}\theta_{t-1} + u_t^\theta, \end{array} \tag{42}$$

we obtain the linear dynamic model

$$\begin{array}{l} Y_t = y_t + F\theta_t + e_t,\ e_t \sim N(0, I\sigma), \\ \theta_t = x_t + G\theta_t + u_t,\ u_t \sim N(0, W), \\ \text{where } y_t = A + B_M M_t, \\ x_t = \mu_\theta + \phi_{\theta M} M_{t-1}, \\ F = B_\theta, G = \phi_{\theta\theta}. \end{array} \tag{43}$$

that can be estimated as follows:

$$\begin{array}{l}
\text{Given: } (\theta_{t-1}|D_{t-1}) \sim N(m_{t-1}, C_{t-1}). \\
\text{Prior: } (\theta_t|D_{t-1}) \sim N(a_t, R_t), \\
\text{where } a_t = Gm_{t-1} \quad R_t = GC_{t-1}G^T + W. \\
\text{Forecast: } (Y_t|D_{t-1}) \sim N(f_t, Q_t), \\
\text{where } f_t = Fa_t \quad Q_t = FR_tF^T + \sigma. \\
\text{Posteriori: } (\theta_t|D_t) \sim N(m_t, C_t), \\
\text{where } m_t = a_t + A_t(Y_t - f_t), C_t = R_t - A_tQ_tA_t^T, A_t = R_tFQ_t^{-1}.
\end{array} \tag{44}$$

Once the conditional distribution of $(\theta_t|D_t)$ $t = 1..T$ is obtained, the FFBS algorithm permits one to obtain a sample of $(\theta_t|D_T)$.

$$\begin{array}{l}
\text{Given } (\theta_T|D_T) \sim N(m_T, C_T). \\
(\theta_t|\theta_{t+1}) \sim N(h_t, H_t), \\
\text{where } h_t = m_t + B_t(\theta_{t+1} - a_{t+1}) \quad H_t = C_t - B_tR_{t+1}B_t^T \quad B_t = C_tG^TR_{t+1}^{-1}.
\end{array} \tag{45}$$

A.2 IRF and Variance Decomposition: continuous time version

The time impulse response function in discrete time is

$$X_t = \Sigma\varepsilon_t + \Phi\Sigma\varepsilon_{t-1} + \Phi^2\Sigma\varepsilon_{t-2} + \Phi^3\Sigma\varepsilon_{t-3} + ... \tag{46}$$

When $Y = A + BX$, clearly the response of the shocks in the yield curve should be

$$\begin{array}{ccccc} B\Sigma\varepsilon_t & B\Phi\Sigma\varepsilon_t & B\Phi^2\Sigma\varepsilon_t & B\Phi^3\Sigma\varepsilon_t & ... \\ t+0 & t+1 & t+2 & t+3 & ... \end{array}. \tag{47}$$

In continuous time, we have

$$X_{t_i|t_{i-k}} = e^{-K(t_i - t_{i-k})}X_{i-k} + \sum_{l=0}^{k-1} \int_{t_{i-k+l}}^{t_{i-k+l+1}} e^{-K(t_i-u)}\Sigma dw_u. \tag{48}$$

Using the approximation (17), the response of X_t to a shock ε_t in a interval of time of $dt = 1/252$ (one day) turns out to be .

$$\begin{array}{ccccc} \Sigma\sqrt{dt}\varepsilon_t & e^{-Kdt}\Sigma\sqrt{dt}\varepsilon_t & e^{-2Kdt}\Sigma\sqrt{dt}\varepsilon_t & e^{-3Kdt}\Sigma\sqrt{dt}\varepsilon_t & ... \\ t+0 & t+1 & t+2 & t+3 & ... \end{array} \tag{49}$$

similarly

$$\begin{array}{ccccc} B\Sigma\sqrt{dt}\varepsilon_t & Be^{-Kdt}\Sigma\sqrt{dt}\varepsilon_t & Be^{-2Kdt}\Sigma\sqrt{dt}\varepsilon_t & Be^{-3Kdt}\Sigma\sqrt{dt}\varepsilon_t & ... \\ t+0 & t+1 & t+2 & t+3 & ... \end{array} \tag{50}$$

is the response of Y_t. In discrete time, the Mean Squared Error of the s-periods ahead error $X_{t+s} - EX_{t+s|t}$ is

$$MSE = \Sigma\Sigma^{\intercal} + \Phi\Sigma\Sigma^{\intercal}\Phi^{\intercal} + \Phi^2\Sigma\Sigma^{\intercal}(\Phi^2)^{\intercal} + ... + \Phi^s\Sigma\Sigma^{\intercal}(\Phi^s)^{\intercal}. \tag{51}$$

The contribution of the j-th factor to the MSE of X_{t+s} will be then

$$\Sigma_j\Sigma_j^{\intercal} + \Phi\Sigma_j\Sigma_j^{\intercal}\Phi^{\intercal} + \Phi^2\Sigma_j\Sigma_j^{\intercal}(\Phi^2)^{\intercal} + ... + \Phi^s\Sigma_j\Sigma_j^{\intercal}(\Phi^s)^{\intercal}. \tag{52}$$

The j-th factor contribution to the MSE of Y_{t+s} is

$$B\Sigma_j\Sigma_j^T B^{\intercal} + B\Phi\Sigma_j\Sigma_j^{\intercal}\Phi^{\intercal}B^{\intercal} + B\Phi^2\Sigma_j\Sigma_j^{\intercal}(\Phi^2)^{\intercal}B^{\intercal} + ... + B\Phi^3\Sigma_j\Sigma_j^{\intercal}(\Phi^3)^{\intercal}B^{\intercal}. \tag{53}$$

In continuous time, it turns out that the s-period ahead MSE of is an integral:

$$MSE = \int_t^{t+s} e^{-K(t+s-u)}\Sigma\Sigma^{\intercal}(e^{-K(t+s-u)})^{\intercal}dt \tag{54}$$

Hence, the contribution corresponding to the j-th factor in the variance decomposition of X_{t+s} and Y_{t+s} at time t are

$$\begin{array}{c} \int_t^{t+s} e^{-K(t+s-u)}\Sigma_j\Sigma_j^{\intercal}(e^{-K(t+s-u)})^{\intercal}dt, \\ B^{\intercal}\left(\int_t^{t+s} e^{-K(t+s-u)}\Sigma_j\Sigma_j^{\intercal}(e^{-K(t+s-u)})^{\intercal}dt\right)B. \end{array} \tag{55}$$

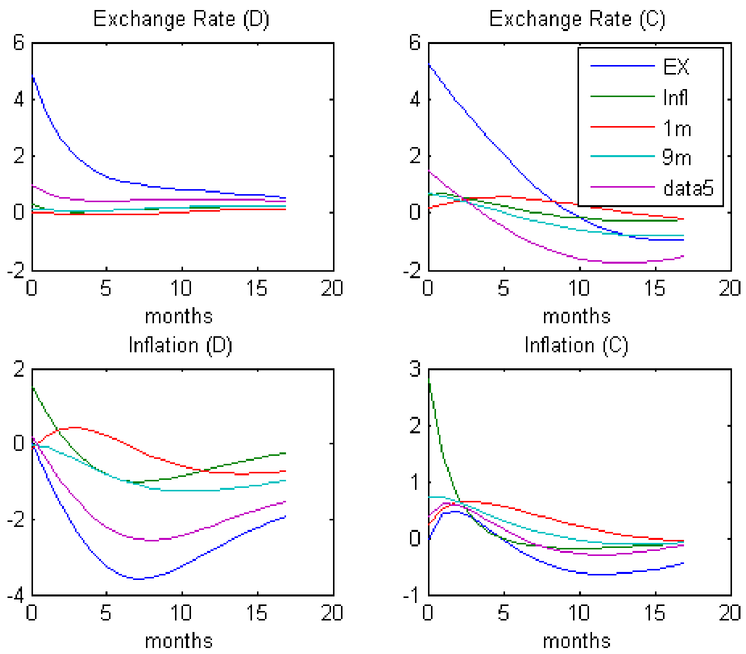

Figure 1: Impulse Reponse Functions of the 2 latent variables plus expected inflatin and exchange rate model. Macro shocks.

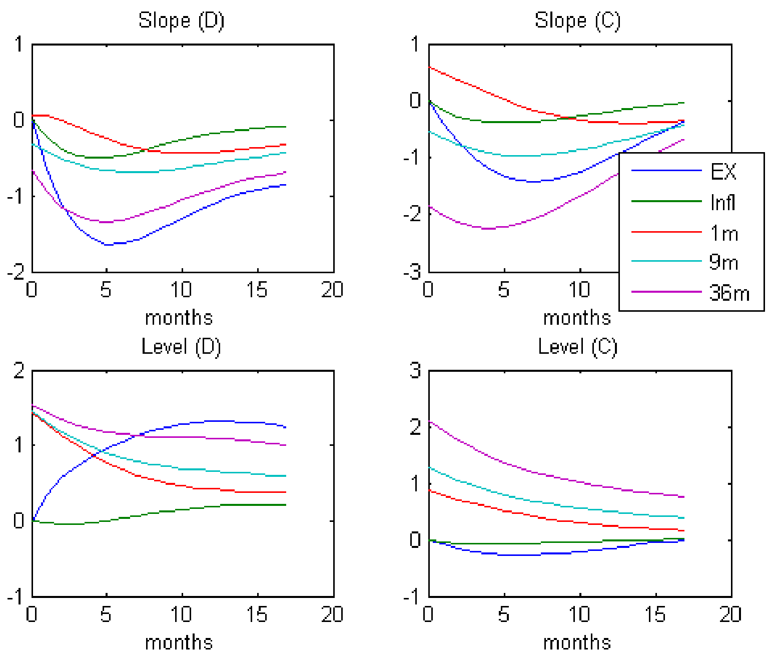

Figure 2: Factor shocks.

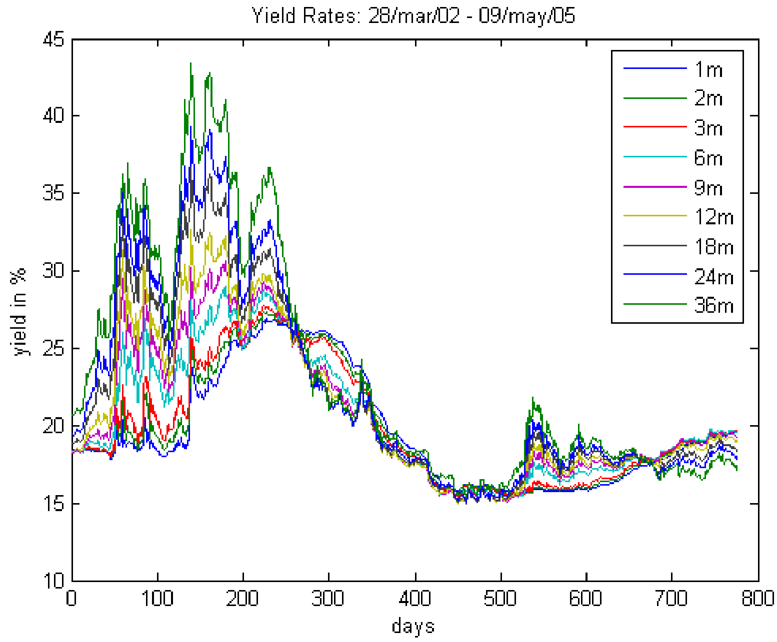

Figure 3: Evolution of the Brazilian Domestic Term Structure (April 2002, October 2005).

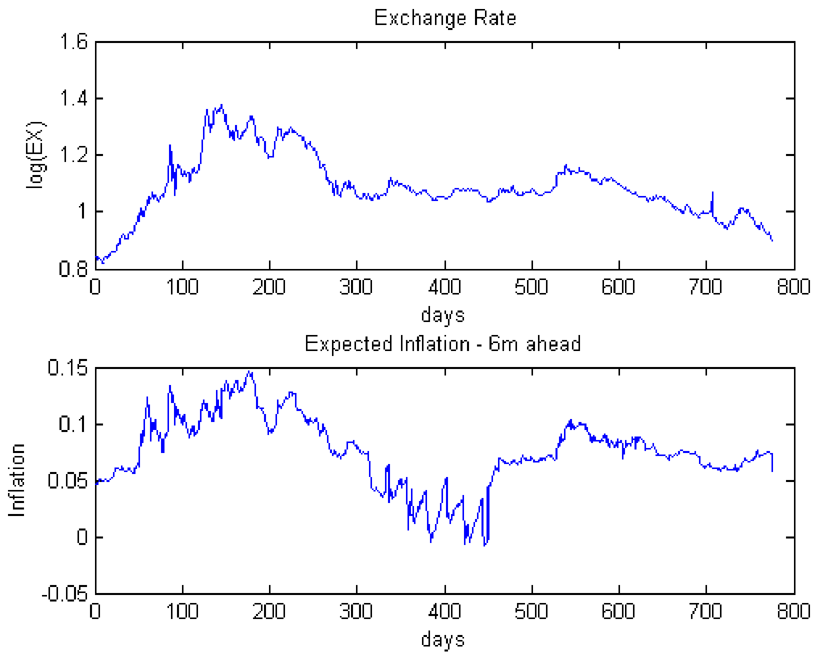

Figure 4:

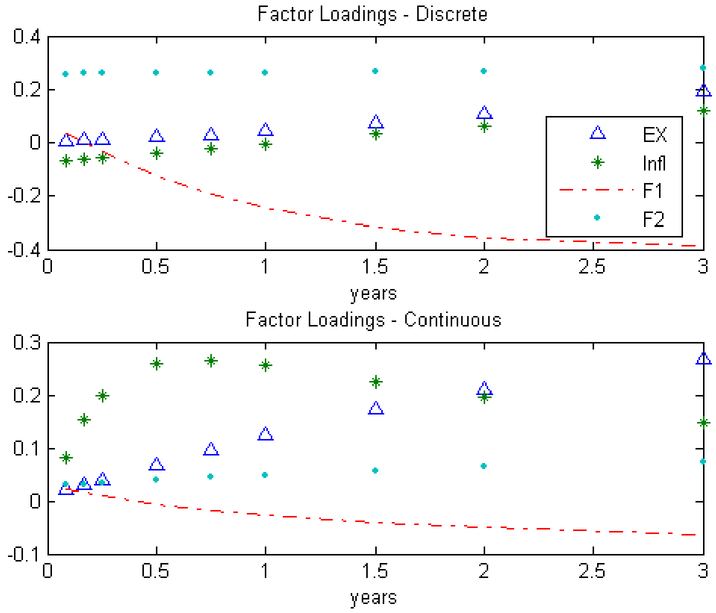

Figure 5:

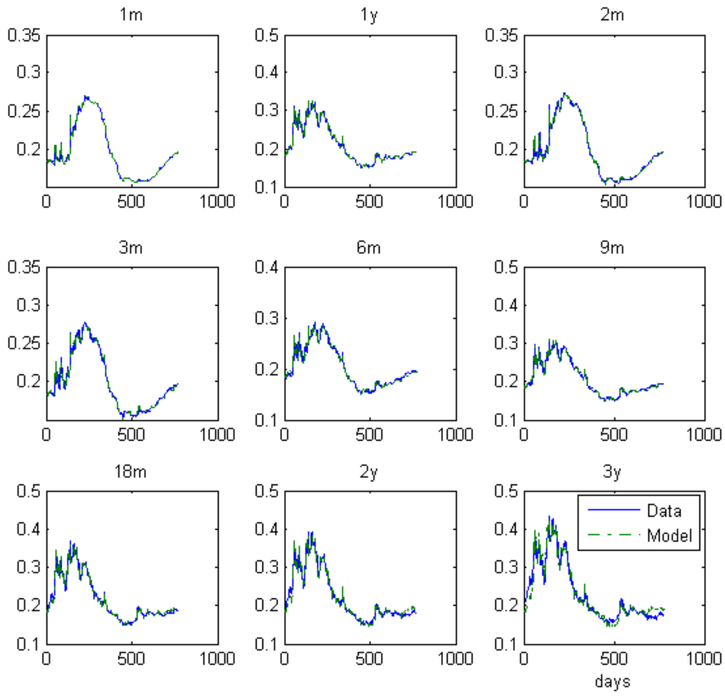

Figure 6: Fitting of the model.

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