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PETROLEUM CONCESSIONS WITH EXTENDIBLE OPTIONS: INVESTMENT TIMING AND VALUE USING MEAN REVERSION AND JUMP PROCESSES FOR OIL PRICES*

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SUMÁRIO

RESUMO

O detentor da concessão de exploração de petróleo no Brasil tem uma opção de investimento até uma data de expiração fixada pela agência governamental, a qual pode estender mediante um custo adicional. O valor desses direitos e a política ótima de investimento são calculados resolvendo um problema de controle ótimo estocástico de uma opção americana de compra com maturidades estendíveis. A incerteza do preço do óleo é modelada como um processo de difusão misto (reversão à média + saltos). Informações normais geram um processo contínuo de reversão à média para o preço do óleo, porém choques aleatórios produzem saltos discretos e estocásticos. Comparações com o tradicional movimento geométrico browniano são realizadas bem como quantificações e análises de políticas alternativas ótimas para o setor de petróleo.

ABSTRACT

The owner of a petroleum exploration concession in Brazil has an investment option until the expiration date fixed by the governmental agency, which can be extended by additional cost. The value of these rights and the optimal investment timing are calculated by solving a stochastic optimal control problem of an American call option with extendible maturities. The uncertainty of the oil prices is modeled as a mix diffusion-jump process. Normal information arrival generates continuous mean-reverting process for oil prices, whereas a random abnormal information generates a discrete jump of random size. Comparisons are performed with the popular geometric Brownian process and also the quantification and analysis of alternative timing policies for the petroleum sector.

1 - INTRODUCTION

The opening of Brazilian petroleum sector has been attracting several firms, especially in natural gas industry and in exploration and production (E&P) of petroleum. The fiscal regime of Brazilian E&P sector is the concession (lease) contract, which firms offer bonus to Brazilian National Petroleum Agency (ANP) in first-price sealed bidding process. The first bid round is likely to occur in the beginning of 1999, so the details of the concession contract can evolve. In this paper we use a preliminary version for this contract, published by ANP in February/1998, which contains some features such as the possibility of extension for the exploratory period if the concession owner spend some fee to ANP and/or additional exploratory investment. The adequate concession timing policy in the exploratory phase is one of more polemic points of the actual industry debate, which this paper intends to contribute with some quantification of alternative timing policies. This paper focuses the investment considering the scenario described above, under conditions of market uncertainty.

This paper is related to the theory of irreversible investment under uncertainty (real options). As new contribution, we use the framework of options with *extendible maturities*, known before only for financial options,[1] not for real assets. In addition, we use a mix stochastic process (mean-reversion plus jumps) to model the petroleum prices which, despite of its economic logic, has not been used before in petroleum economic literature.

This model is useful for both, firms evaluating concession/investment decisions, and Govern evaluating sectoral policy (mainly the concession time policy) considering that firms have rational expectations and will act optimally when facing investment decisions.

2 - THE CONCESSION AND THE STOCHASTIC MODEL FOR PETROLEUM PRICES

Consider that the concession owner drilled a wildcat well and discovered a petroleum field. After the appraisal phase (delineation of the field) the technical uncertainty about reserves is residual, remaining only market uncertainty driven by the oil price oscillations. Suppose an average size oilfield (100 million barrels in the base case) with a small net present value (NPV). Due to the uncertainty in oil prices, a small down variation of these prices can transform this positive NPV into a negative one. So, can be optimal for the firm to wait for better market conditions (delaying this positive NPV project), even paying a fee to extend the concession period. However, for a sufficiently high NPV (when the option is “deep in the money”), the cost of delaying the operational cash flows is larger than the benefit

[1] For a discussion on extendible maturities options, see Longstaff (1990) and Briys et al. (1998, chapter 16). The payoff of an extendible option is the maximum of two risky payoffs: the payoff from a standard call option and a compound option (call on a call) less the cost to get it. So, extendible options are more general than compound options.

of waiting so that is optimal the immediate oilfield development even with large timing freedom. The model presented here identifies the optimal investment rule and oilfield value.

In exploratory phase, considerations about technical uncertainty (existence, size and quality of petroleum reserves) are very important. The integration of this development decision model (market uncertainty oriented) with exploratory decisions features (additional technical uncertainty issues) can be performed easily with simple models,[2] but also with more complex models.[3] Even for the exploratory decision, this development valuation is necessary in order to evaluate the exploratory prospect itself. In the spirit of dynamic programming the calculations are performed backwards, so we need to know the terminal value (option to develop a delineated oilfield) in order to estimate the initial value (exploratory prospect with some probability to find out an oilfield). So, development models are also useful (and a necessary step) for a good exploratory valuation model.

Oil price, the main source of market uncertainty, is modeled with a special stochastic process: a combined diffusion-jump process. This model follows Merton's (1976) concept on asset prices oscillations. The arrival of normal information over an infinitesimal time interval generates only marginal adjustment of the prices, which is modeled by a continuous diffusion process, whereas the arrival of abnormal information (very important news) generates a discrete stochastic shock (jump), which is modeled as a Poisson process. This combination is also named Poisson-Gaussian model.

The adopted diffusion process for petroleum prices is the *mean-reversion process* because it is considered the natural process choice for commodities.[4] Normal information mainly means smoothly or marginal interaction between production versus demand (inventories is an indicator) and depletion versus new reserves discoveries (the ratio reserves/production is an indicator). Basic microeconomics theory tells that, in the long run, the price of a commodity ought to be tied to its long-run marginal production cost or, "in case of a cartelized commodity like oil, the long-run profit-maximizing price sought by cartel managers" [Laughton & Jacoby (1995, p. 188)]. Production cost varies largely across the countries (mainly due to the geologic features) and most of the lower cost countries belong or are influenced by the Opec cartel. Hence, even with a growing non-Opec production, the Opec rôle remains very important in the production-price game of the petroleum industry.

[2] See Dias (1997) for a simple model integrating three kind of uncertainties (technical, economic and strategic) using an option-game framework with a compact decision-tree plus a game-tree for the exploratory phase.

[3] See Dixit & Pindyck (1994, Section 4, chapter 10) book for a continuous time model combining both technical and market uncertainty. Although the model is drawn to nuclear industry, it can be adapted for the petroleum one.

[4] See for example Pilipovic (1998, Table 4-9, p. 78): her test showed that the best model for WTI petroleum is the log of Price mean-reverting.

In other words, although the oil prices have sensible short-term oscillations, it tends to revert back to a "normal" *long-term equilibrium* level. Pindyck & Rubinfeld (1991, chapter 15) using a Dickey-Fuller unit root test, show that the oil price reversion to a long-run equilibrium level is likely to be slow, and rejected the random walk hypothesis (geometric Brownian motion) for oil prices in case of very long-term series (more than 100 years).

Other important mean-reverting evidence comes from *futures market*, as pointed out by Baker, Mayfield and Parsons (1998, p. 124-127). First, the *term structure* of futures prices are decreasing (toward the "normal" long-run level, in *backwardation*) if the spot prices are "high", and are increasing (even in *contango*) if prices are "low". Second, if the prices are random walk, the *volatility* in the futures prices should equal the volatility of the spot price, but the data show that spot prices are much more volatile than futures price. In both cases, the mean-reverting model is much more consistent with the futures prices data than random walk model. In addition, the econometric tests from futures term structure performed by Bessembinder et al. (1995, p. 373-374) also reveals strong mean-reversion for oil prices and agricultural commodities (but weak reversion for precious metals and financial assets).

The Poisson-jump[5] can be either positive or negative for petroleum prices, depending of the kind of economic/politic abnormal news. In petroleum history there were abnormal news causing large jumps in petroleum prices, along few weeks. For example: jumps in 1973/74 (Iom Kipur war and Arabian oil embargo), in 1979/80 (Iran revolution and Iran-Iraq war), in 1986 (Saudi Arabia price war), in 1990 (Kuwait invasion by Iraq) and in 1991 (the Iraq defect). At least three large jump-ups and two jump-downs for oil prices can be identified in these events. This feature is incorporated into the model, which allows either direction for the jumps and a stochastic size for the jump. We follow Merton (1976), except that he used log-normal distribution for the jumps size instead of the two truncated-normal distribution that we assumed, and he used geometric Brownian[6] instead of mean reversion for the continuous process. The mean-reverting+jump model was used before for interest rate [see Das (1998, p. 4)], but despite its economic logic, was not used before for oil prices.[7]

[5] In real options literature, most jump processes has been used to model random competitors arrival [see Trigeorgis (1996, p. 273; 284-288; 328-329)], and in outcomes from R&D projects [see Pennings & Lint (1997)].

[6] Merton models financial stock prices (not commodities), which geometric Brownian looks more appropriated than mean-reversion [see Bessembinder et al. (1995, p. 373-374)]. In general, commodities (except precious metals like gold, which have financial assets characteristics) and interest rates are best represented by mean-reversion.

[7] Dixit & Pindyck (1994, chapter 12, Section 1.C) point the difficulties to determine the "correct" stochastic process for oil price. Although they don't suggest explicitly the jump+mean-reversion together, they indicate separately both processes as possible good models for oil prices and the importance for oil firms to take these features into account.

Let P be the spot price of one oil barrel. Most of times the prices change continuously as a mean-reverting process and sometimes the oil prices change discretely following a Poisson distribution. This petroleum prices mean-reverting + jump model has the following stochastic equation:

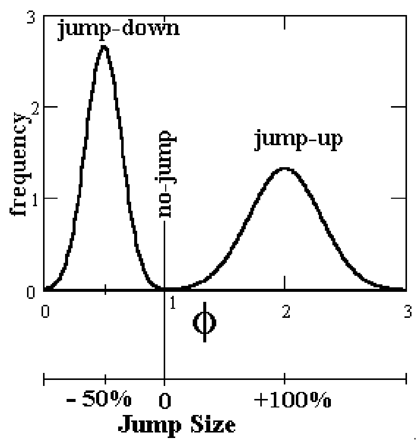
$$\frac{dP}{P} = [\ (\bar{P} - P) - \quad k]\ dt + \quad dz + dq \qquad (1)$$

$$dq = \begin{cases} 0, \text{ with probability } 1 - \lambda dt \\ \phi - 1, \text{ with probability } \lambda dt \end{cases}$$

$$k = E(\phi - 1)$$

Equation (1) for the rate of variation of petroleum prices (dP/P) has three terms in the right side. The first one is the mean-reverting drift: the petroleum price has a tendency to go back to the long-run equilibrium mean $\bar{P}$ with a reversion speed η. The second term presents the continuous time uncertainty represented by volatility σ, where dz is the Wiener increment. The last term is the jump one, with the Poisson arrival parameter λ (there is a probability $\lambda.dt$ to occur a discrete jump). The jump has random size: ϕ has a special probability distribution with mean $k+1$, represented by two truncated-normal distributions, one normal distribution for the jump-up and the other one for the jump-down, as illustrated bellow.

Figure 1
Random Jumps Distribution



Given a jump event, this abnormal movement has the same chances to be up or down. The figure shows that, in case of jump-up, the price is expected to double, whereas in case of jump-down the price is expected to drop to the half (we assume large expected jumps but with low frequency, of 0.15 per annum). However, the exact size of each jump is uncertain.

By taking expectations in equation (1), is easy to see that $E(dP/P) = \eta\ (\overline{P} - P)\ dt$, that is, the process is expected to revert, and this tendency is higher as far the current price is from the long run mean. This is like an elastic/spring force. In this process is useful the concept of *half-life* of the oil price process, which is a more practical alternative measure for reversion speed (or slowness to revert), bringing a time idea for prices going to the long-run prices. Half-life (H) is the time that is expected the oil price to reach the intermediate value between the current price and the long-run average price, and is given by the following equation (see the Appendix A for the proof and additional information).

$$H = \frac{\ln(2)}{\eta \overline{P}} \qquad (2)$$

The diffusion–jump model has economic logic appeal and a good mapping for the probability distribution along the time for the oil prices. The model presents complex empirical problems due to the additional parameters estimation, when comparing with a more popular and simpler models, the geometric Brownian motion (GBM). However, GMB models are less rigorous than the diffusion-jump stochastic process presented above, and this disadvantage can be important to model long-maturities options like undeveloped reserves. Other important models for oil prices are the two and three factors models, and models with stochastic long run price, that we discuss briefly.

The two-factor model [Gibson & Schwartz (1990)] generally uses GBM for spot oil prices and a stochastic mean-reverting convenience yield δ. This additional factor corrects the main bias from the one-factor GBM model, becoming more consistent with the market data from the futures term structure. The three-factor model is presented in the interesting article of Schwartz (1997*a*, p. 929-931), allowing the interest rate to be the third stochastic factor, also modeled as mean-reverting. The three stochastic processes are correlated, there is a complex empirical job (using Kalman filter) to estimate several parameters of these processes, and he compares it with one and two-factor models.

Another important class of models allows the equilibrium long-term price level to be stochastic, presented in Baker et al. (1998, p. 134-135) and in Schwartz & Smith (1997). Both papers argue that this model is equivalent to the two-factor stochastic convenience yield model. The model has economic logic because is likely that the equilibrium level changes with the evolution of variables like the marginal cost for price takers producers, the correlation of forces between Opec and non-Opec, new environmental regulations, new technologies, politic scenario, etc. The equilibrium price is likely to be positively correlated with spot prices. In our model this equilibrium price is assumed constant, which is reasonable in the context because our stochastic process describes the oil prices only until the exercise of the option (or the expiration), assuming a market value of reserves after the exercise. For models that describe also the cash flow after the option exercise, this assumption could be a necessary improvement (see discussion later).

3 - THE TIMING OF INVESTMENT AND THE OPTIMIZATION PROBLEM

Let the instant $t = T_1$ be the primary (or the first) expiration of the concession option. At this time the owner has three alternatives: to develop the field immediately, to pay a fee (and/or additional exploratory investment) to extend the maturity of the option (looking for better conditions to invest), or to give up the concession, returning the tract to the ANP. So the firm has, in addition to the classic option model with the decision for the maximum between NPV and zero, the decision to buy another option paying the fee. Let the instant $t = T_2$ be the second and definitive expiration of the concession option. At this time the firm will choose the maximum between NPV and zero. This second expiration is like the classic option case. We consider that the operating project value $W(P)$, that is, the project value *after* the investment, can be conveniently given by the following equation:

$$W(P) = B \,.\, V(P) = B.\, q\, P \qquad (3)$$

Where B is the quantity of barrels of reserves in the ground (the reserve volume) and V is the market value of one barrel of reserve. We assume that this value is proportional to oil prices, which has been used as assumption in real options models [see Paddock et al. (1988) and Dixit & Pindyck (1994, chapter 12, Section 1)]. Consequently, V follow the same stochastic process of P. The proportion factor q is, in average, 33% of oil price ("one-third" rule of the thumb), but can be a different proportion for different cases of reserves. This proportion is named *economic quality of a developed reserve,*[8] because the higher is q higher is the operational profit from this underlying asset.

The value of q is assumed constant and independent of the price, which could be view as one critical assumption for "pure reversion thinking". But we argue with the observed high positive correlation between V and P[9] and the value of q itself can be estimated using the expected oil prices trend from a mean-reverting model

[8] Dias developed this concept in a Petrobras course on real options. See more details in his speech at Stavanger Workshop on Real Options (May 1998) at http: //www. nitg.tno. nl/dss/Public/public_ activities.html

[9] The main reference for the market value of reserve, published by the traditional John S. Herold since 1946 [see data and discussion in Adelman et al. (1989, mainly Table 2)], shows a large positive correlation between P and V, including jumps. Examples: between 1981/85, V was in the range of 8-10 \$/bbl, whereas in 1986 dropped to 5.88 \$/bbl; for the 70's oil prices shocks, jumps in V were still more pronounced. The volatility of V has been slight lower than P.

or using information from futures market (decreasing the bias) as in Schwartz (1997*a*, equation 18 or equation 30). In addition, due the effect of depletion and discounting, the operating cash flows from the first five years has higher importance in the reserve value than distant cash flows.

Schwartz (1997*a*, p. 971) results give us another important argument, using very different stochastic models that driven heavily on futures markets insights (the two and three-factor models mentioned before). These models imply an underlying *project value that is linear with the spot price* [Schwartz (1997*a*, Fig. 13)], the inclination of his two or three factor NPV is exactly our economic quality q, and hence can be reproduced with our equation (3). This contrasts with the predictability of the "pure reversion model" that undervalues the project in high spot prices scenario and overvalues the project in the low price case. In practice, the simplification of q constant corrects some bias from "pure reversion".[10]

If we consider the extension of our stochastic process for the time horizon of the operating cash flows (that is, after the exercise of the development option), would be necessary to consider additional features. For a more complete model could be important allow for the operational options (expansion or speed up, temporary stopping, abandon) and, perhaps more relevant, improving our stochastic process by allowing the long run equilibrium price to be stochastic instead constant. These upgrade features are left to a further work, but based in the above reasons, we think that our error is not much important to justify going deeper by now. For example, in the high price case by taking account the option to speed up production (with additional wells or early production systems),[11] we get some offsetting effect over the expected reduction in V due to the expected price reversion. Managers periodically can also revise the value of q to be used in equation (3), so that for the low spot price case, an revised/slight-increased value of q (calculated using the new cash flow expectations) can lower the threshold of the investment, permitting the (more realistic) optimal exercise.

We going to work in values per-barrel (of course is also possible to work in total values), so afterwards we use NPV to express this value per barrel, so:

$$NPV \;=\; V(P) - D \;=\; q\,P - D \qquad (4)$$

where D is the development investment per barrel of reserve.[12]

[10] The Schwartz's models assume that the operational costs (*OC*) are deterministic and independent of the commodity price P. However, for oilfields, the correlation between *OC* and P has been very high as shown the data from Adelman et al. (1989, Table 2). By the other side, our model simplifies assuming perfect correlation. The truth is in between. A more realistic but more complex model should allow for stochastic costs with a positive correlation with P process.

[11] Early Production Systems were exactly that happen in Brazil in the high prices times from early 80's.

[12] So, an investment of US$ 200 millions in a 100 million barrels oilfield means D = US$ 2/bbl.

The investment in our model even being non-stochastic, in the first period (0-T_1) can be different from the second period (T_1-T_2) value. For example, suppose the extension fee (K) is an additional exploratory well. If this well could be used as a development well (as producer or as injector), the extension investment can be reduced by a certain quantity due to this well use.[13] So we use D_1 for the investment (per barrel) until the first expiration and D_2 for the investment in the extension period ($D_1 \geq D_2$). If the additional exploratory well is a good investment independently of the extension benefit, is possible to consider the traditional option model (instead of extendible options) with a single maturity at T_2 (because the additional exploratory cost will be done anyway).

We want to find out both the value of the concession (the value of the option to invest) $F(P,\ t)$, and the optimal decision rule thresholds (the decisions are: to invest, or to wait, or to extend the option, or even to give up). The solution procedure can be view as a maximization problem under uncertainty. We use the Bellman-dynamic programming framework [see Dixit & Pindyck (1994, chapter 4)] to solve the stochastic optimal control problem. We want to maximize the value of the concession option $F(P,\ t)$ seeking the instant when the price reach a level P^* (the threshold) in which is optimal one type of action (investment or pay to extension). The Bellman equations are:

$$F_1(P,t) \equiv \max_{P_1\ *(t)} \begin{Bmatrix} [V(P) - D_1,\ E[F_1(P + dP,\ t + dt)\, e^{-\rho dt}],\ \text{for all}\ t < T_1 \\ [V(P) - D_1,\ E[F_2(P + dP,\ t + dt)\, e^{-\rho \text{dt}} - K,\ 0],\ \text{for}\ t = T_1 \end{Bmatrix} \qquad (5)$$

$$F_2(P,t) \equiv \max_{P_2\ *(t)} \begin{Bmatrix} [V(P) - D_2,\ E[F_2(P + dP,\ t + dt)\, e^{-\rho dt}],\ \text{for all}\ T_1 < t < T_2 \\ [V(P) - D_2,\ 0],\ \text{for}\ t = T_2 \end{Bmatrix} \qquad (6)$$

Where ρ is an exogenous discount rate, that can be a CAPM[14] like risk-adjusted discount rate for the underlying asset if the market is sufficiently complete, or an arbitrary exogenous discount rate in case of incomplete markets. For complete markets is also possible to use "risk-neutral" valuation, by using a risk-free interest rate instead ρ, but in this case is necessary to change the drift of the stochastic process.[15] The risk-neutral approach relies in the absence of arbitrage opportunities or dynamically complete markets. [For a discussion of dynamic programming versus risk neutral approach (and the *contingent claim* approach), see Dixit & Pindyck (1994, chapter 4).

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[13] In Brazil frequently an exploratory well is used in the development project. Even if the well is not the better location for the project, the investment reduction due to the already drilled well can be a good compensation. Kemna (1993, based in her consultant for Shell) presented a model for extendible options, but not allow for any benefit derived from the fee/additional exploratory extendible cost. She developed a simplified model, using European style option.

[14] Capital Asset Pricing Model, a mean-variance equilibrium model, is used to set discount rates for assets and projects.

[15] The equivalent alternative (largely used in derivatives pricing) is a probability transformation, using artificial probability (or *martingale* measure) instead of the real probability process.

Let us consider a more general assumption in the model: the jump-risk is *systematic* (correlated with the market portfolio) so it is not possible to build a riskless portfolio,[16] or the market is not complete for this model with non-diversified jump risk. The other alternative for incomplete markets models is a more restrictive assumption, using single-agent optimality framework and/or detailed equilibrium description, as performed in Naik & Lee (1990) for jumps in the market portfolio itself. This more complex approach needs to specify the investor utility. In petroleum corporations there are hundreds of thousands of stockholders, with different levels of wealth and so with different utilities. So, a complex approach trying to specify utility has practical disadvantage, without to be much more rigorous than the adopted dynamic programming model as Dixit & Pindyck, using an exogenous (e.g. corporate rate) or a "market-estimated" discount rate ρ (for details see Section 4.1).

We are interested in find out the optimal path $P_1*(t \leq T_1)$, $P^E(T_1)$ and $P_2*(T_1 < t \leq T_2)$, as well as the value of concession $F(P, t)$ in each of these periods. Using the Bellman equation and the Itô's Lemma, is possible to build the following partial differential-difference equation (PDE):

$$\tfrac{1}{2}\,\sigma^2 P^2 F_{PP} + \{\eta(\overline{P} - P) - \lambda\, E[\phi - 1]\}\, P\, F_P + F_t + \lambda\, E[F(P\, \phi, t) - F(P, t)] = \rho F \qquad (7)$$

With the following boundary conditions:

$$F(0, t) = 0 \qquad (8)$$

$$F_1(P, T_1) = \max\, [V(P) - D_1, F_2(P, t) - K, 0] \qquad (9)$$

$$F_i\,(P^*, t) = V(P^*) - D_i \qquad (10)$$

$$F_2\,(P, T_2) = \max\, [V(P) - D_2, 0] \qquad (11)$$

$$F_P\,(P^*, t) = V_P(P^*) = q \qquad (12)$$

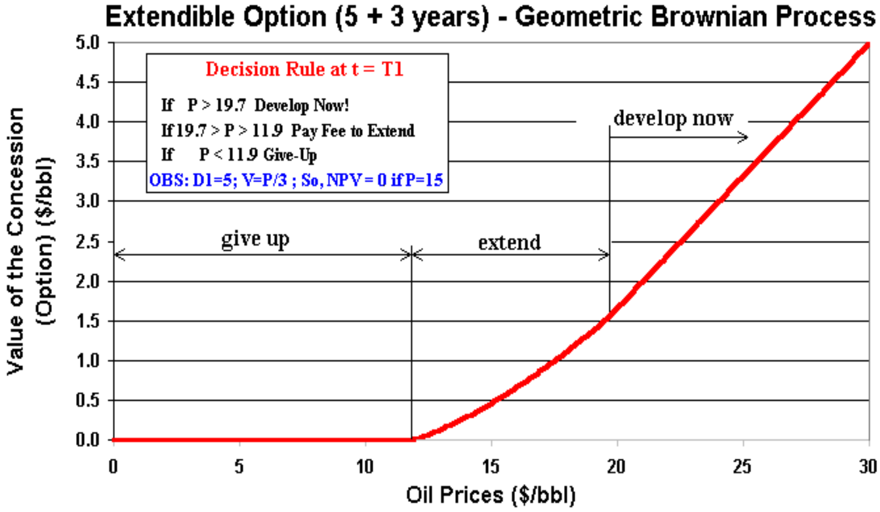
The equation (7) is a PDE of parabolic type and is solved using the numerical method of finite differences in the explicit form (see Appendix B). A C++ program with a graphical interface (see Appendix D) was developed to solve this model and to perform the comparative statics analysis.

[16] For oil prices, is hard to say if jumps are or not really systematic, but theoretically a jump in the oil demand by the market (mainly in crashes/recessions case) could cause a price jump. This is a good topic for a further research. See the Trautmann & Beinert (1995, Section 3.2) discussion of systematic jumps and the empirical evidence for stocks. Nietert (1997, p. 1-4) distinguishes firm-specific, industry-specific (with systematic component) and market jumps for stocks.

The boundary conditions [equation (8) to equation (12)] are typical for an American call option with extendible maturities. The second condition [equation (9)], for the first expiration T_1, means to choose between the alternatives: to develop, to extend and to give up (respectively in the max. parenthesis). The lowest price at T_1 that we choose to extend the option paying K is the *extension threshold* P^E. The last equation (12), known as "smooth pasting condition", is equivalent to the optimum exercise condition, so alternatively can be performed the earlier exercise test (the maximum between the lived option and the payoff $V - D$).

Figure 2 shows the extendible option at the first expiration moment ($t = T_1$) identifying the three possible range of petroleum prices associated to different decisions (give up, extend or develop now) at the first expiration. The threshold values are also displayed in the chart. This graph is close to traditional option payoff chart, except for the region between 11.9 and 19.7 US$/bbl, where the optimal action is to extend the option (see the curve with option shape for the interval where is optimal to extend the option). This graph is typical for the geometric Brownian motion (GBM), and the shape is similar to the presented in the mentioned paper of Longstaff (1990, Fig. 1, p. 939).

Figure 2
Extendible Option at the First Expiration (for Brownian Motion Base Case)



4 - COMPARATIVE STATICS

4.1 - Base Case: The Parameters

The Table 1 shows the parameters values used for the base case. Some values were estimated using available data about oil prices and/or using available related literature, such as the volatility, the long-run average oil price, the reversion speed, the jump size and jump frequency. Others were assumed as representative values for Brazilian offshore oilfields, such as the investment at both expirations, the cost to extend the option and the economic quality of the reserves. The assumed times to expiration consider international practice.[17] The (per-barrel) investment cost, the current spot oil prices and the economic quality of the reserves are set so that in the base case the NPV of the project is zero. The extension cost, of US$ 0.3/bbl means US$ 30 million for a 100 million barrels of reserve, which is approximately the cost of two deepwater exploratory wells. A preliminary empirical job to estimate the parameters for the jump-reversion stochastic process using oil prices time series (mainly for the volatility and the reversion speed) is shown in the Appendix C, that used market data from the Brent oil, the main oil reference in Europe. Comparison of this jump-reversion process base case with the popular geometric Brownian motion is presented in Section 4.3.

Table 1

Parameters from the Base Case for Jump+Mean-Reverting Model

Parameter	Notation	Base Case Value
Volatility of the Diffusion Process (% p.a.)	σ	22
Exogenous Discount Rate (% p.a.)	ρ	10
Reversion Speed ; [Half-Life (years)]	η ; [H]	0.03 ; [1.16]
Annual Frequency of Jumps (per annum)	λ	0.15
Economic Quality of Developed Reserve	q	0.333
Long-Run Average Oil Price (US$/bbl)	$\overline{P}$	20
Average Jump-Up (%)	μ_u	100
Standard Deviation of the Jump-Up (%)	s_u	30
Average Jump-Down (%)	μ_d	– 50
Standard Deviation of the Jump-Down (%)	s_d	15
First Expiration (years)	T_1	5
Second Expiration (years)	T_2	8
Investment up to T1 (US$/bbl)	D_1	5
Investment after T1 until T2 (US$/bbl)	D_2	4.85
Cost to Extend the Option (US$/bbl)	K	0.3

The long run equilibrium price is hard to obtain. One reference is a long run Opec price goal of about US$ 21/bbl, but the long run marginal cost from non-Opec countries, under US$ 19/bbl, could be used as lower bound. The increasing non-Opec production has been offsetting by the rising costs experimented by oil

[17] The preliminary version of the concession contract in Brazilian pointed out three years plus two years of extension, but we believe this timing will be enlarged as most oil companies wish. We indicate some reasons for this.

companies going to deepwater and ultra-deepwater to find new reserves. Perhaps the best value is in between the Opec and the non-Opec marginal cost.[18]

Baker et al. (1998, p. 129) estimate of the long run oil price was $18.86/bbl (in 1995 dollars) and used (p. 138-140) $20/bbl as initial long run level in their model.[19] In the same article, one graph (p. 127) of term structure of futures prices suggests a long-run price between 18-21 $/bbl. We adopt $20/bbl (in 1998 dollars) for the Brent crude. This value is also adopted in Bradley (1998, p. 59-61) and shown in Cortazar & Schwartz (1996, Figure 4). Our $\bar{P}$ value is constant along the option term.

For the *half-life* value, although in general the values from literature are higher (two+ years) than our estimate, the values that Bessembinder et al. (1995, p. 373-374) found in futures market (data from March 1983 to December 1991) are very close to ours. Extrapolating the values from their Table IV, we find an implicit half-life of 1.1 years, practically the same of ours estimate (1.16 years). Bradley (1998, p. 59) also finds a half-life close to our base case (of 1.39 years).

We assume an exogenous discount rate ρ of 10% p.a. (which is also the official discount rate to report the present value of proved reserves to stock market investors) in the base case. In reality, with our general assumption of systematic jump risk, is not possible to use the non-arbitrage way to build a riskless portfolio because market information is not sufficient to spawn *all* the risk. In this case there is no theory for setting the "correct" discount rate (CAPM doesn't hold), unless we make restrictive assumptions about investors' utility functions (without guarantee of more reliable results).

One practical "market-way" to estimate ρ is looking the net convenience yield (δ) time series (calculated by using futures market data from longest maturity contract with liquidity),[20] together with spot prices series, estimating ρ by using the equation: $\rho = \delta + \eta(\bar{P} - P)$. Here δ in general is just the difference between the discount rate (total required return) and the expected capital gain $E(dP/P)$, like a dividend. The parameter δ is endogenous in our model and, from a market point of

[18] A suggestion for further research is to set the long-run equilibrium price modeling with the game theory, seeking a Nash equilibrium. Even duopoly models (Opec and eventual allied versus price takers producers) could be interesting. Pure statistical approach could be noisy, misleading the evolving forces correlation between the players.

[19] That paper uses an uncertain long run equilibrium price modeled with geometric Brownian model, with this price growing exponentially. Other important model for oil prices is the "two-factor" model, with uncertain convenience yield (modeled as mean-reverting) and geometric Brownian model for the oil prices [e.g. Gibson & Schwartz (1990)].

[20] The known formula for a commodity futures prices is $F(t) = e^{(r - \delta)\, t} P$. This equation is deducted by arbitrage and assumes that δ is deterministic, so it looks contradictory with our assumption of systematic jump and with our model that implies that δ is as uncertain as P. But we want an implicit value for δ and so for ρ, to get a market reference (a bound) to set ρ. It is only a practical "market evaluation" for the discount rate that is assumed constant in our model. This futures market approach to get the δ series is used even to estimate parameters for models with stochastic δ.

view, is used in the sense of Schwartz (1997*b*, p. 2) description: "*In practice, the convenience yield is the adjustment needed in the drift of the spot price process to properly price existing futures prices*". High oil prices P in general means high convenience yield δ (positive correlation),[21] and for very low P the net convenience yield can be even negative. There is an offsetting effect in the equation (even being not perfect), so we claim as reasonable the approximation of ρ constant. In compensation, we don't need to assume constant interest rate (because it doesn't appear in our model) or constant convenience yield (here implicitly changes with P).[22] The series (P, δ) permits to estimate an *average* "market" ρ (from the ρ time series that we will get with this approach) or by looking the intercept from the simple regression P x δ. In this way, the value of ρ depends heavily of the assumed values for η and $\overline{P}$. This is only a bound for ρ in the general model.

The alternative, using the same market data, is to estimate the return ρ on this commodity by running a cointegrating regression of the temporal series (P, δ) or by estimating the risk premium running a simple regression of futures and spot prices [see Pindyck (1993, p. 514-517)].[23]

Figure 3 shows the option value for the base case at the current data (t = 0, upper/thinner line) and the payoff line (bottom line) at the first expiration (T_1). The option curve shape is different of the Brownian motion case, the option graph exhibits a typical shape for mean-reverting process. See the option curve smooth pasting on the payoff line: the tangency point is the threshold for immediate investment. The main thresholds of the base case are showed in the chart.

Figure 4 displays the payoff lines for the two options expirations for the base-case. The thinner (black) line is for the first expiration whereas the other one (green) is for the second expiration. Note that the options have different exercise prices (D_1 and D_2).

Figure 5 shows the options value at the first expiration (payoff, bottom line) and the option curve (upper line) just after the first expiration, for a case slight different of the base-case (with higher fee to extension, K = 0.5 \$/bbl, in order to highlight the effect). Note that the payoff and the option curve are parallel in the

[21] Schwartz (1997*a*, p. 943, Table IX) finds strong correlation between the spot price and the convenience yield (+0.915 for 259 samples and + 0.809 for other 163 samples). The correlation between spot price and interest rate (r) were slight negative (–0.0293 and –0.0057), whereas between δ and r seem to be independent (-0.0039 and +0.0399).

[22] Much less realistic is the GBM assumption of δ constant. Even the superior two-factor model of Gibson & Schwartz (1990) needs to assume that both the interest rate and the market price of convenience yield risk (p. 967) are constant.

[23] The Pindyck (1993) model is based in the "fundamentals" (present value of δ stream), so is also not strictly coherent with our model with systematic jumps, but again his suggested market way to estimate ρ could be a good reference.

Figure 3
Extendible Option at t = 0 and at the First Expiration (Mean-Reverting + Jump)

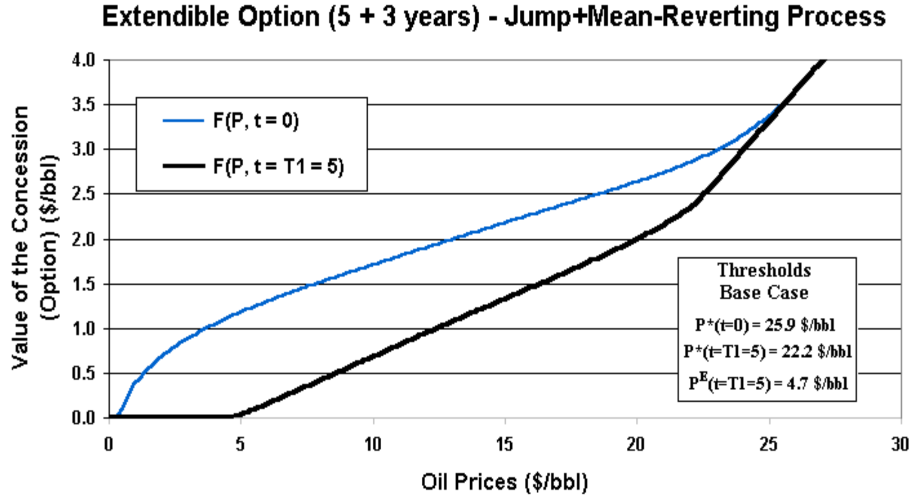
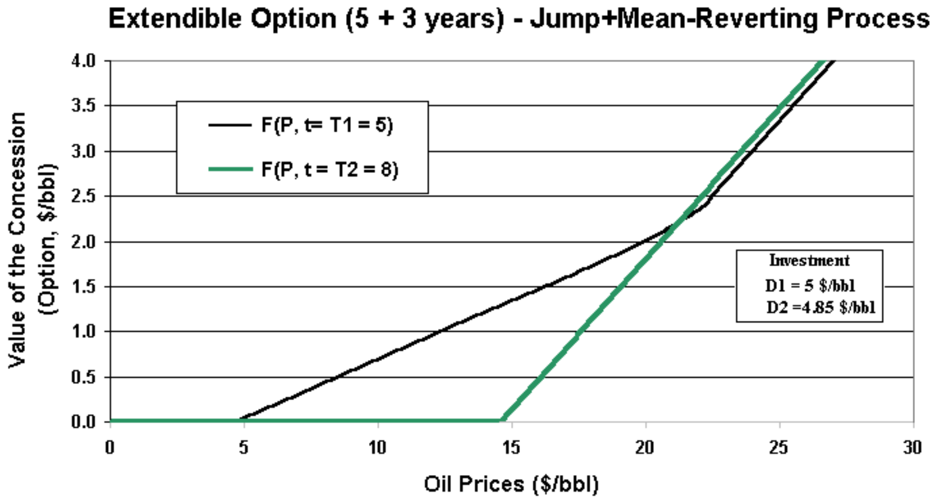


Figure 4
The Payoff Lines for the Options Expirations (Base Case)



interval that is optimal to extend the option and also that the distance between the parallel lines is K, the fee to be paid in order to extend the option.

Figure 5
The Options Value at the First and Just after the First Expirations (for $K = 0.5$)

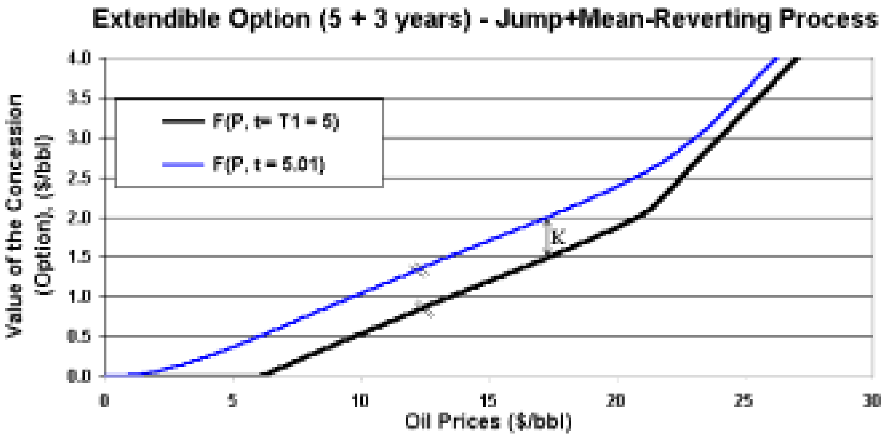
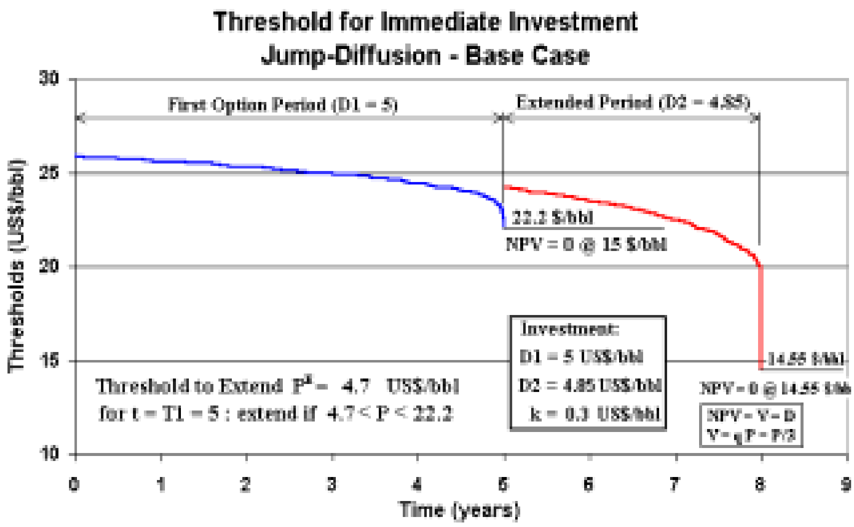


Figure 6 displays the threshold lines for both terms of the option. On or above the threshold lines, is optimal the immediate investment.

Figure 6
The Threshold Lines for Jump-Diffusion Process (Base Case)



4.2 - Sensibility Analysis

Several sensibility analysis were performed for each parameter of this process. Some parameters show higher impact on both option value and thresholds than other. For example, the economic quality of the developed reserves (q) has large impact, mainly in the option value: from q = 0.2 to 0.45, the option $F(P$=15) rises from 0.46 to 4.00 $/bbl and the threshold $P^*(t$=0) drops from 30.9 to 24.4 $/bbl. However, for the standard deviations of the jump size (both up and down) simulations have shown a minor impact on the results.

One interesting analysis in this transition phase of Brazilian petroleum sector is about time to expiration policy. The Table 2 shows that an increase in the time to expiration has major impact over the option value than over the threshold. The table points that rising the total expiration (T_1+T_2) from five years to eight years, the option value increases near 20%, whereas the threshold value increases less than 4%. So, eight years instead five years attract higher bid bonus[24] (~ proportional to option value) without delaying (looking the thresholds) too much good investment projects.

In the base case, the option value reach US$ 2.178/bbl, which is significantly higher than NPV value (NPV is zero for P = 15 $/bbl). For a 100 million barrels oilfield, this mean a value of US$ 217.8 million.[25]

Table 2
Sensibility of the Time to Expiration Value for Option and Threshold

T_1 (years)	T_1 + T_2 (years)	$F(P$=15) ($/bbl)	% in F	$P^*(0)$ ($/bbl)	% in P^*
2	3	1.440	-	24.1	-
3	5	1.828	26.9	25.1	4.1
5	8 (base case)	2.178	19.2	25.9	3.2
6	10	2.314	6.2	26.2	1.2
8	12	2.417	4.5	26.4	0.8

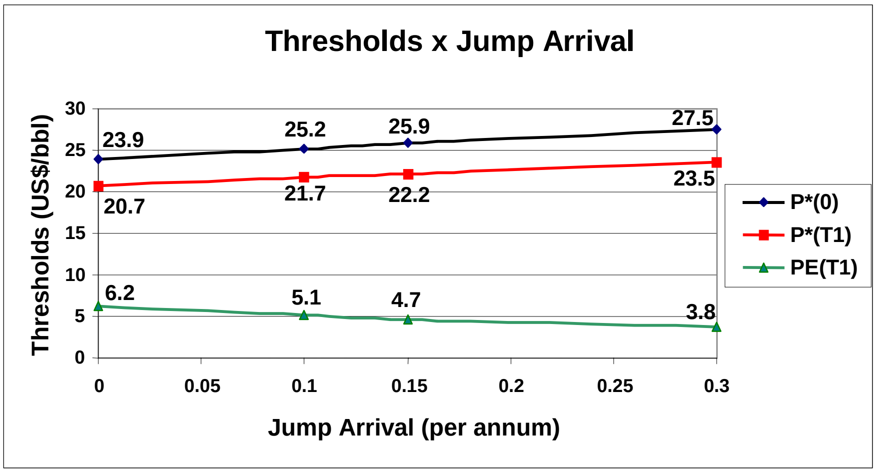
[24] The ANP President estimated in R$ 600 million (a little more than US$ 500 million) the expected bonus for the 1999 concession sales (Brazilian press interview in 11/10/98). In USA Gulf of Mexico lease sales, the govern agency (MMS) has been collecting more than US$ 1 billion/year in the last years (even in 1998). Although the geologic potential in deepwaters be lower than in Brazil Campos Basin, the MMS economic freedom policy justify these observed high bonus. The MMS policy for deepwaters leases includes 10 years timing, royalty relief for about 80 million barrels cumulative production, and the sale of small blocks (that attracts small companies and so there are more bidders per block, implying higher bonus).

[25] The ANP President speculated an undiscovered reserve of 22.5 billion barrels in Brazil (Brazilian press interview in 11/10/98). If our base case can be considered representative, and if the above potential is confirmed by exploration drilling, this means an option value around US$ 50 billion (exploratory costs not included, but the base case of oilfields with NPV = 0 for P = 15 is conservative in Brazil).

Moreover, higher time to expiration presents other benefits (so higher bonus-bid) that were not considered in this paper. For example: *a*) "Bayesian" gain of sequential exploratory investment (rather parallel) using information gathered for correlated prospects; *b*) low attractiveness for one firm to bid several tracts, if the time is too small to perform optimal sequential investment (according the "auction theory", less bidders per tract means lower expected bonus value); *c*) an economic /optimal planning of resources (e.g. deepwater rigs) allocation is damaged if the timing is too short, losing business opportunities that are available on specific timing like seasonal rates of special service ships, etc.; and *d*) *revelation* of exploratory work in the basin [see Dias (1997, p. 143)] that reduces technical uncertainty and points out new geologic plays currently not considered (which leverage the tract value and so the winner bid, if there is time to wait and to use this information).

Figure 7 shows the thresholds sensibility with the Poisson arrival factor λ. For higher jump frequency the threshold level for the immediate investment is higher, which has economic logic because the investor is less willing to invest due to the risk of jump-down. However the threshold for the extension decreases, because jump-up increase the possibility of a not good project to transform into a good one. Hence, in most cases, firms should pay a small cost to extend the option rights.

Figure 7
Thresholds and the Arrival Jump Frequency λ



The comparative statics results in general were: option values increased for higher reversion speed, lower discount rate,[26] higher volatility, higher jump arrival, higher jump-up mean, lower extension cost, higher long-run mean, higher economic quality of the reserve, and higher expiration time.

4.3 - Comparing Geometric Brownian with Jump+Mean-Reversion

Geometric Brownian Motion (GBM) also known as drifted random walk model, is the most popular stochastic process and is generally a very good, stochastic process in financial economics, although far from perfect, mainly for commodities. The GBM model for the oil prices is shown bellow.

$$dP = \alpha P \, dt + \sigma P \, dz \qquad (13)$$

where $\alpha = \rho - \delta$ and dz is the Wiener increment.

The parameters for GBM base case are $r = \delta = 5\%$, $\sigma = 23\%$. The comparison of the GBM with our more rigorous jump-diffusion model [equation (1)] for oil prices is summarized in the Table 3.

Table 3
Option Values at P = US$ 18.3/bbl

Jump + Mean Reversion Process: $F(P = 18.3$ \$/bbl, $t = 0)$					
Base	No-Jump ($\lambda = 0$)	No volatility $\sigma = 0\%$	$\sigma = 5\%$	No reversion $\eta = 0$	$\sigma = 23\%$, $\lambda = 0$ and $\eta = 0$
2.4768	1.8979	2.0225	2.2592	1.8237	1.4162

Geometric Brownian Motion: $F(P = 18.3$ \$/bbl, $t = 0)$		
Base ($r = \delta = 5\%$)	$r = 10\%$ and $\delta = 5\%$	$r = 10\%$ and $\delta = 10\%$
1.5739	2.0831	1.4162

The table was built with a convenience yield δ of 5% for both processes. In the case of GBM, δ is a parameter input of the model, and is constant. In the case of Jump+Mean-Reversion, δ is not constant, is not a direct parameter input (it is implicit, endogenous of the model) and depends of the price level: $\delta(P)$. In order

[26] Increasing the discount rate ρ, decrease both the option and the threshold at $t = 0$ because, given a fixed drift, the convenience (dividend) yield δ has to adjust to the chances in ρ due to the relation $\rho = \eta(\overline{P} - P) + \delta$. Increasing the convenience yield, the waiting value decreases and so the threshold and the option value. See Dixit & Pindyck (1994, chapter 5) for further explanation of mean-reversion process and sensibility analysis for the discount rate.

to compare in the same basis, let us choose a petroleum price to compare options so that dividend yield is 5%. This price is 18.3 US$/bbl because this implicit convenience (or dividend) yield for jump-mean-reverting process means:

$$\delta = \rho - \eta(\overline{P} - P) \Rightarrow 0.05 = 0.1 - 0.03(20 - P) \Rightarrow P = 18.3 \text{ \$/bbl}$$

Comparing jump+mean-reversion and GBM (Table 3), jump+mean-reversion in general presents higher option values. The GBM has higher option value only for higher interest rate case (r = 10%, the same value of ρ in jump-diffusion setting) *and* when comparing with no jumps (λ = 0), no volatility (σ = 0) or no reversion (η = 0) cases. The option value is closer of GBM in case low uncertainty (5%) for the reversion process. However, the rôle of interest rate r in the GBM and the ρ in the jump-diffusion are very different. The option value increases with r in the GBM and decreases with ρ in the jump-diffusion (see last footnote). In the GBM r is independent of δ, so the only effect is to increase the waiting benefit (imagine the investment amount is in the bank earning r), but ρ is not independent of δ in the jump-diffusion model. In other words, for the same drift $\eta(\overline{P} - P)$ a change in the value of ρ, implicitly means change in the value of δ. For this reason, if we use a lower value for ρ (e.g. ρ = 5% = r), we get a higher option value (2.5814, not shown in the table), and the option values from jump-diffusion process become still higher than GBM. For η = 0, implying $\rho = \delta$, we can compare the case of GBM with $r = \delta$ and jump-diffusion for no reversion, no jump and with the same volatility (η = 0, λ = 0, σ = 23%). In this case, as expected, the values are the same, equal to 1.4162 (see Table 3). For a jump-diffusion with ρ = 5%, and also with σ = 23%, η = 0, λ = 0 (not shown in table), the option is again the same of the GBM base case (which has $r = \delta$ = 5%), that is, 1.5739.

Figure 8 shows both thresholds, for the jump-diffusion process and for the GBM. The threshold curve is smoother for GBM than for jump-diffusion process near expirations. The reason is the *effect of the dividend-yield*. In case of GBM, the dividend δ is constant and positive, whereas for jump-diffusion process, δ is not constant (depends of oil prices P). In jump-diffusion process δ is positive for higher oil prices and negative for lower prices. A well known property from American options is that earlier exercise only can be optimal if $\delta > 0$. So, earlier exercise is possible only if P is higher than 16.7 $/bbl (in the base-case) and this explain the discontinuity of the threshold curve at the expiration.[27]

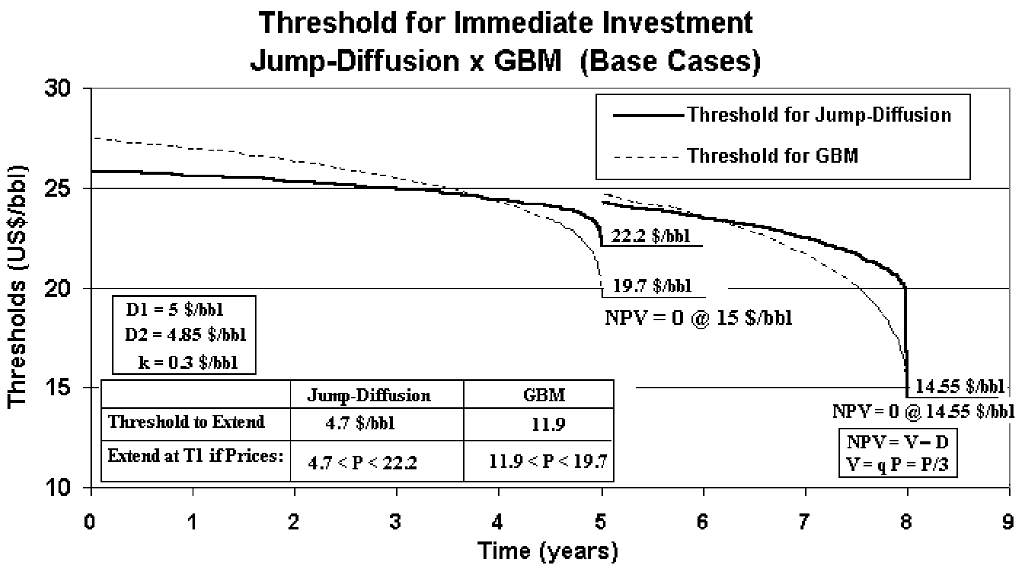
The other important observation is that the threshold to undertake the project in the beginning of the term is higher for the GBM (although in general the option

[27] At the expiration ("now-or-never") the option is the maximum between NPV and zero. NPV is zero for P = 14.55 $/bbl (= threshold at expiration $T2$). In the threshold curve there is a gap because a minute before the expiration, a *necessary* condition to exist an optimal exercise is P > 16.7 (in order to get $\delta > 0$) and δ is *sufficiently* positive to optimal earlier exercise only at around the level of $20/bbl.

values are lower). This is coherent with the results of Schwartz (1997*a*, p. 972)[28] when comparing GBM with their two and three-factor models even being very different from our jump-diffusion model, but that systematically produce results qualitatively close of our model when he compares these models with the GBM.

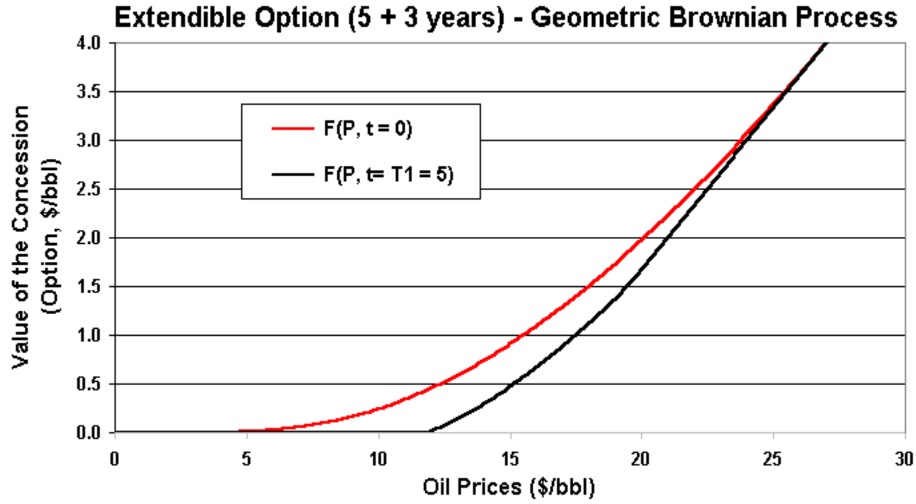
Figure 9 presents the main option value curves for the first period (at $t = 0$ and $t = T_1$). See the smooth pasting property at the curve tangency, indicating the optimal earlier exercise of the option.

Figure 8
Thresholds Comparison: Jump-Diffusion x Geometric Brownian Motion



28 Schwartz (1997*a*, equation 52 and footnote 35) compares thresholds using perpetual options for the GBM and 10 years maturity for the two and three factors models. However for the volatility used, the threshold for perpetual and 10 years maturity are very close (threshold asymptotic property for long term), permitting the comparison.

Figure 9
Extendible Option Using Geometric Brownian Motion



5 - CONCLUSIONS

This paper develops a model for extendible options embedded in the offshore oil contracts in several countries, that is also presented in the preliminary version of Brazilian offshore leases contract for petroleum exploration and development decision. The model incorporate the possibility of the extension cost (e.g. exploratory wells) to be used partially to reduce the development cost.

Sensibility analysis of the parameters, suggest a higher option value (and so expected higher bonus bid) to a higher time to expiration without a significant additional delay of investment in good projects. So a suggestion from these results is to consider higher times to expiration than the first values pointed by ANP in the preliminary concession contract. Moreover, there are other benefits (so even higher bonus-bid) from higher time to expiration that was not quantified in this paper.

The stochastic model of jump+mean-reversion for the oil prices has more economic logic than previous models used in real options literature, considering that normal news causes continuous small mean-reverting adjustment in oil prices, whereas abnormal news causes abnormal movements in these prices (jumps). A future improvement is to allow for stochastic long run equilibrium price (mainly for longer terms), calculating the initial equilibrium price (of the industry players) by the game theory.

The comparison of this more rigorous model with the more popular Geometric Brownian Motion pointed a higher option value for the jump-diffusion case. Hence, a higher expected bid in the lease-sale process is a consequence of using this more rigorous model. Other good models from literature like Gibson & Schwartz (1990) and Schwartz (1997a) two and three-factor, that rely more heavily on the futures markets, despite being very different, present results qualitatively very similar with ours.

Several extensions are possible for our main model. For example: *a*) allowing the equilibrium price level to be stochastic; *b*) using a correlated stochastic process for the operational cost, instead the adopted linear function *V*(*P*); *c*) incorporating the technical uncertainty and exploratory revelation; *d*) considering other options like sequential development (extendible call on a call) and/or abandon (extendible call on a put); and *e*) portfolio planning, quantifying the expected first hitting time for a project that currently is optimal to wait, in order to estimate when the investment is expected to start.

Appendixes

Appendix A
Mean Reversion and Half-Life

The model of oil prices reversion known as Geometric Ornstein-Uhlenbeck is used in Dixit & Pindyck (1994) and in Metcalf & Hasset (1995):

$$\frac{dp}{P} = \eta\,(\bar{P} - P)\,dt + \sigma\,dz \qquad \text{(A1)}$$

This model (A1) has the same forecasting expected value (A3) of our actual jump-diffusion model (A2):

$$\frac{dP}{P} = [\eta\,(\bar{P} - P) - \lambda k]\,dt + \sigma\,dz + dq \qquad \text{(A2)}$$

$$E\,(\frac{dP}{P}) = \eta\,(\bar{P} - P)\,dt \qquad \text{(A3)}$$

We define *half-life* (H) of the petroleum prices[29] as the time for the expected oil prices to reach the intermediate (middle) price between the current price and the long run mean. This oil price half-life is deducted below:

From equation A3: $dP/[P\,(\bar{P} - P)] = \eta\,dt$

Integrating from $P_0\,(t_0)$ to $P_1(t_1)$, and letting $\Delta t = t_1 - t_0$, we get:

$$(1/\bar{P})\left[\ln\left(\frac{-1}{-P + \bar{P}}\right)\right]_{P_0}^{P_1} = \eta\,\Delta t$$

$$\Rightarrow\ \ln\left(\frac{P_1 - \bar{P}}{P_0 - \bar{P}}\right) = -\eta\,\bar{P}\,\Delta t \qquad \text{(A4)}$$

For Δt = half-life H, by definition we have that $(P_1 - \bar{P}) = 0.5\,(P_0 - \bar{P})$, hence:

[29] The original concept comes from the physics: measuring the rate of decay of a particular substance, half-life is the time taken by a given amount of the substance to decay to half its mass.

$$\ln(0.5) = -\eta \bar{P} \mathrm{H} \Rightarrow -\ln(2) = -\eta \bar{P} \mathrm{H}$$

$$\Rightarrow \boxed{\mathrm{H} = \frac{\ln(2)}{\eta \bar{P}}} \tag{A5}$$

From (A4) we can get the expected oil price at the generic instant t_1:

$$E(P_1) = \bar{P} + (P_0 - \bar{P})\, e^{-\bar{P} \eta \Delta t} \tag{A6}$$

In others papers appeared a slightly different half-life equation: $H' = \ln(2)/\kappa$, where κ is a reversion speed. This equation comes from models like in Smith & McCardle (1997). They model the *logarithm* of the oil prices as mean-reverting, $\pi = \ln(P)$, with the Ornstein-Uhlenbeck process:

$$d\pi = \kappa(\bar{\pi} - \pi)\, dt + \sigma\, dz$$

Following the same procedure above, is easy to show that the half-life of this process is $H' = \ln(2)/\kappa$. This logarithm model has some advantages (for example the long-run mean doesn't appear in the half-life equation), but the half-life is for the logarithm of prices and not for the prices itself.

Our model has the practical advantage of the half-life interpretation: instead entering with reversion-speed or half-life of $\ln(P)$, which has small intuitive appeal, we enter with oil prices half-life that has a more intuitive/managerial interpretation: as reversion parameters the user (manager) enter the long-run average and the number of years which is expected the oil price to reach the half distance towards the long run mean.

The difference between the two equations can be small (but not negligible). For example, if the long-run mean is US$ 20/bbl and the current oil price is US$ 15/bbl, when we say *price half-life* of two years we mean that in two years the oil price will reach US$ 17.5/bbl (not necessary a calculator to find out). For the same data, when people say two years for the *logarithm of price half-life* they mean that in two years the oil prices will reach US$ 17.3/bbl (with the help of a calculator due to the logarithms).

Besides this point, people could ask what is the mean-reverting model that fits better with oil prices data. This is an interesting but complex empirical job, which is beyond the paper scope.

Appendix B
Explicit Finite Difference Numerical Solution

To solve the partial differential equation (PDE) of parabolic type we use the finite difference method (FDM) in the explicit form. It consists of transforming the continuo domain of P and t state variables by a network or mesh of discrete points. The PDE is converted into a set of finite difference equations which can be solved iteratively using the appropriated boundary conditions ($t = T_1$ and $t = T_2$) and proceeding backwards through small intervals ΔPs until we find the optimal path $P^*(t)$ to every t.

Suppose the following discretization for two variables:

$$F(P,t) \equiv F(i\Delta P, j\Delta t) \equiv Fi,j$$

where $0 \le i \le m$ and $0 \le j \le n_1$ or n_2.

With this discretization, we are able to build a rectangular grid and find the solution in it. The grid for P and t variables are $P \in [0, P_{\max}]$ and $t \in [0, T_2]$, where $P_{\max} = m.\Delta P$.

The boundary conditions say that the variable P assumes an infinite value ($P_{\max} = \infty$), which is captured in the model doing the m value the biggest possible. As T_1 and T_2 (end of each expiration phase) are known, n_1 and n_2 are given by $n_1 = T_1 / \Delta t$ and $n_2 = T_2 / \Delta t$.

The choice of the discrete steps must be done in a way that all the coefficients of the finite difference equation be always positive to any value inside the grid to ensure the convergence of explicit FDM. So the convergence of the FDM settles the choice of ΔP and Δt.

The partial derivatives are approximated by the following differences:

$$F_{PP} \approx [F_{i+1.j} - 2F_{i,j} + F_{i-1,j}] / (\Delta P)^2 \; ; \; F_P \approx [F_{i+1.j} - F_{i-1,j}] / 2\Delta P \; ; \; F_t \approx [F_{i.j+1} - F_{i,j}] / \Delta t$$

We use the "central-difference" approximation for the P variable and the "forward-difference" for the t variable. Applying these approximations to the PDE and its respective boundary conditions we have the following difference equation:

$$F_{i,\, j} = p^{+} F_{i+1,\, j-1} + p^{0} F_{i,\, j-1} + p^{-} F_{i-1,\, j-1} + p_{jump} E\left[F\left[i.\tilde{\phi},\, j-1 \right] \right]$$

$$p^{+} = \frac{\Delta t}{\Delta t.\rho + 1}\left[\frac{\sigma^2 i^2}{2} + \frac{i.(\eta.\overline{P})}{2} - \frac{i^2.\eta.\Delta P}{2} - \frac{i.\lambda.k}{2}\right] \quad ; \quad p^{0} = \frac{\Delta t}{\Delta t.\rho + 1}\left[\frac{1}{\Delta t} - \sigma^2 i^2 - \lambda\right]$$

$$p^{-} = \frac{\Delta t}{\Delta t.\rho + 1}\left[\frac{\sigma^2 i^2}{2} - \frac{i.(\eta.\overline{P})}{2} + \frac{i^2.\eta.\Delta P}{2} + \frac{i.\lambda.k}{2}\right] \quad ; \quad p_{jump} = \frac{\Delta t}{\Delta t.\rho + 1}\lambda \quad ; \quad k = E\left[\tilde{\phi} - 1\right]$$

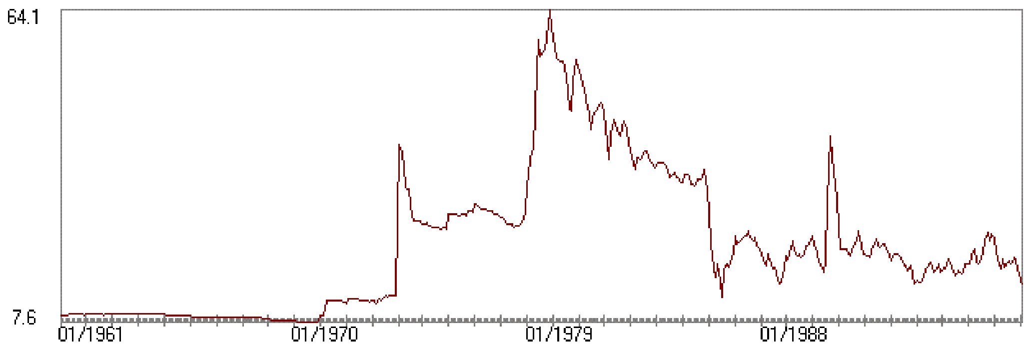
Note that to ensure the convergence of the explicit FDM, we chose • *P* and • *t* in a way that all the *p* coefficients are positives everywhere inside the grid. More about the FDM can be found in Ames (1977), Smith (1971) or Collatz (1966).

Appendix C
Volatility Estimative Using Oil Prices Data

The oil price[30] (see graph bellow for real prices at 1990 US$) suggests that:

- the price of oil is subject to permanent and transitory instabilities;
- the oil price volatility in the beginning of the sample was lower than at the end;
- the coefficients of the model that explains the price based in mean reversion change over history; and
- same if we consider a model that suppose growth rate.

Graph 1



The model instability can be considered on two ways. One, simpler, admits that there was a structural change from a certain moment, say the middle of the 70s, which would recommend discarding the older periods. This view is particularly arbitrary due to the fact that it depends on the cut point and the assessment that only a model change occurred. Alternatively the assessment of a structural change can be incorporated to the model explaining the hypothesis that the parameters follow a random walk.

Even though this last approach is more elegant, it implies in the incorporation of non-linear elements to the model, and in the use of more complex estimation models. Two models will be considered for $(dP/P)_t = p_{t.}$ Model 1 refers to the geometric Brownian motion and Model 2 to the mean reversion process. In both

[30] Light Brent Blend oil (before 1984 were used other similar quality oil from North African, Libya and Qatar). Oil series source: IMF, International Financial Statistics.

models the parameters (δ,ψ) control the adaptability degree of estimations of the pair (a,b), introduce non-linearity and do not allow the finding of analytical forms for its estimates, requiring the use of numerical methods for it.

Model 1: Geometric Brownian Motion

$$p_t = a_t + e_t \qquad e_t \sim N(0, s_t) \qquad \text{(C1)}$$
$$a_t = a_{t-1} + e_{1t} \qquad e_{1t} \sim N(0, \delta s_t)$$

Model 2 admits that the change in oil price in relation to a local average (A) generates tensions in the market that pressure the price toward the average. Since this price is unknown, this representation implies in the product between the speed reversion (b_t) and the medium price (A_t). The model (C2) can be parameterized in the form (C3), where the medium price A = a/b.

Model 2 : Mean Reversion Process:

$$p_t = b_t\,(A_{t-1} - P_{t-1}) + e_t \qquad e_t \sim N(0, s_t)$$
$$p_t = a_t - b_t\,P_{t-1} + e_t \qquad e_t \sim N(0, s_t) \qquad \text{(C3)}$$
$$a_t = a_{t-1} + e_{1t} \qquad e_{1t} \sim N(0, \delta s_t)$$
$$b_t = b_{t-1} + e_{2t} \qquad e_{2t} \sim N(0, \psi s_t)$$

The change in the instability standard of the prices along the sample will be considered using an adaptive model for estimation of volatility(ies). In this specification, the choice of the parameter (θ), which controls the adaptability degree of volatility, is arbitrary. The variance equation s_t follows (C4), with (C5) as solution.

$$s_t = \theta s_{t-1} + (1-\theta)e^2_{t-1} \qquad \text{(C4)}$$
$$s_t = \sum_i \theta^i\, e^2_{t-i} \;/\; (1-\theta) \qquad \text{(C5)}$$

The difference equation (C4) can be solved in the form (C5) that shows the effect of errors of the last (i) periods in the estimation of the (t) period volatility. The value of (i) for (θ^i)=0.5 is denominated half-life of *information* (different of oil prices half-life presented before) and can be used to suggest the relevant values of (θ). The table below shows the estimated values for the volatility of the model (2) — which shows results similar to the ones from model (1) — for many values of (θ), together with their corresponding half-life.

Half-Life (years)	1	2	5	10	∞
θ	0.90	0.95	0.98	0.99	1
Volatility (monthly)	7.11	6.68	6.17	5.75	4.39

The characteristics of prices suggest that the volatility has the same instability behavior since 1979, and because of that we consider relevant the values with half-life smaller than 10 years. For these values the volatility is in the interval [6.17, 7.11], therefore we chose ($\theta = 0.95$) to calculate the estimates.

In this calculation we considered as belonging to the sample the numbers (dP/P) that were in the interval [–0.15, 0.15]. All the others were considered a consequence of the jump. The models (1) and (2) were estimated using the method MCMC (Monte Carlo Markov Chain) — see West and Harrison (1997) — and obtained the after the mode and the interval of maximum density *a posteriori* (IMDP) for 65% level.

	Model 2		Model 1	
	Mode	IMDP(65%)	Mode	IMDP(65%)
δ	.031	[0, .047]	.067	[.022, .136]
ψ	.003	[.001, .006]	-	-
s	6.286	[4.85, 6.62]	6.684	[5.742, 6.72]

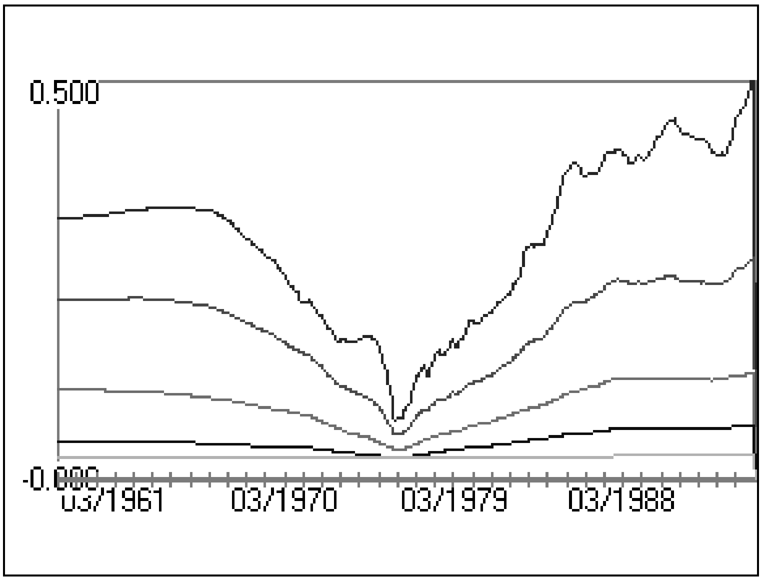
The relevant values for the degree of adaptability of the model are in table above. Each of these corresponds to a possible description for the medium price and the effect of the deviation with respect to the medium price. Using the most likely value of both models we obtain the results bellow.

	Model 2		Model 1	
	Average	Average/S. Deviation	Average	Average/S. Deviation
a	7.08	1.94	-.63	.52
b	.487	2.56	-	-
Medium Price	14.5	-	-	-
Volatility	6.286	-	6.684	-

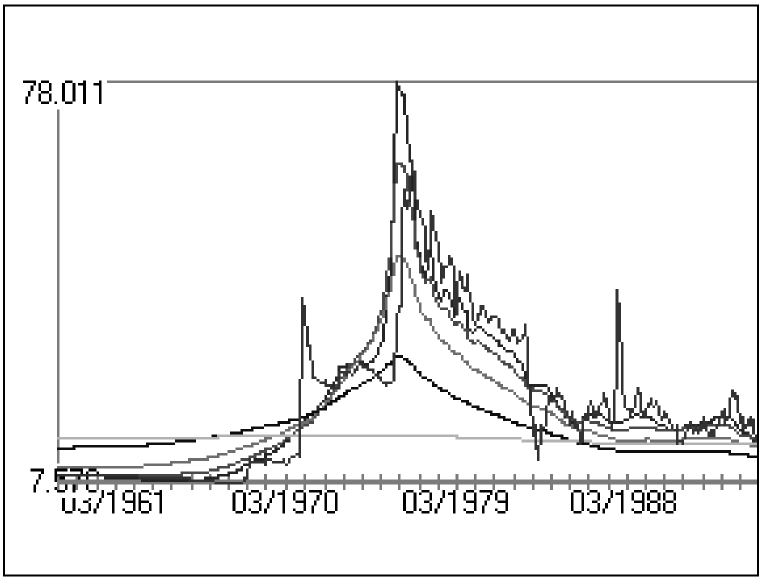
The Graphs 2 and 3 present the results for different adaptability degrees. Note that:

- the speed reversion estimation (b_t) in Graph 2 reduces progressively as long as we reduce the degree of adaptability of the model's parameters (factor 1 to factor 40), leading to a constant that would be obtained with the ordinary least square (OLS) estimate;
- for the most likely results the medium prices (A_t) in the Graph 3 are very close to the spot price (first sharp point curve). As long we reduce the degree of adaptability (factor 1 to factor 40) the medium price tends to a constant which is the result that would be obtained by the model (2) for OLS estimate.

Graph 2
Speed Reversion



Graph 3
Spot Price and Medium Prices



The table bellow presents a summary of the obtained results from the model (2) with different combinations of medium prices and the deviation effect obtained (reversion speed).

Factor*	δ, ψ	Volatility (monthly)	Reversion Speed	Medium Price
1	0.04,0.004	6.50	.50	14.7
8	0.005,0.0005	6.68	.07	12.3
40	0.001,0.0001	6.72	.03	14.4

**Factor 1 indicates the most adaptive model, and so the one which gives more weight to the recent observations, and Factor 40 represents the less adaptive one with almost the same weight to all observations.*

The table above used all data since 1960. One can argue that has been an structural broken in the generating process of data, cause the best adaptive model estimated one medium price of approximate 15, price viewed as too low by the market. Therefore, we can think of discarding the series beginning and repeat the exercise up to 1979, second petroleum shock. The table bellow presents the estimations of data up to 1979.

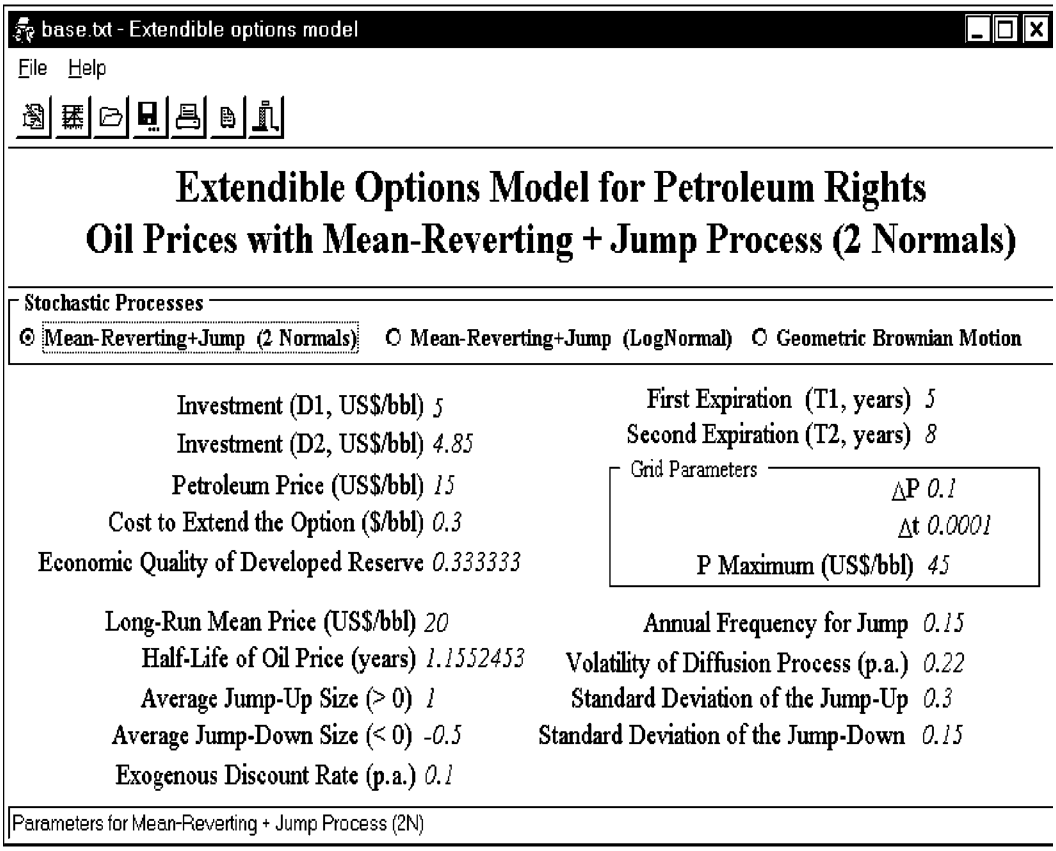
Factor	δ, ψ	Volatility (monthly)	Reversion Speed	Medium Price
1	0.04, 0.004	6.50	.72	16
8	0.005, 0.0005	6.73	.13	17.8
40	0.001, 0.0001	6.77	.043	19.7

Every alternative belongs to the interval of maximum density a posteriori (IMDP) to the level of 65%, therefore your choice can be realized using non-statistical approaches.

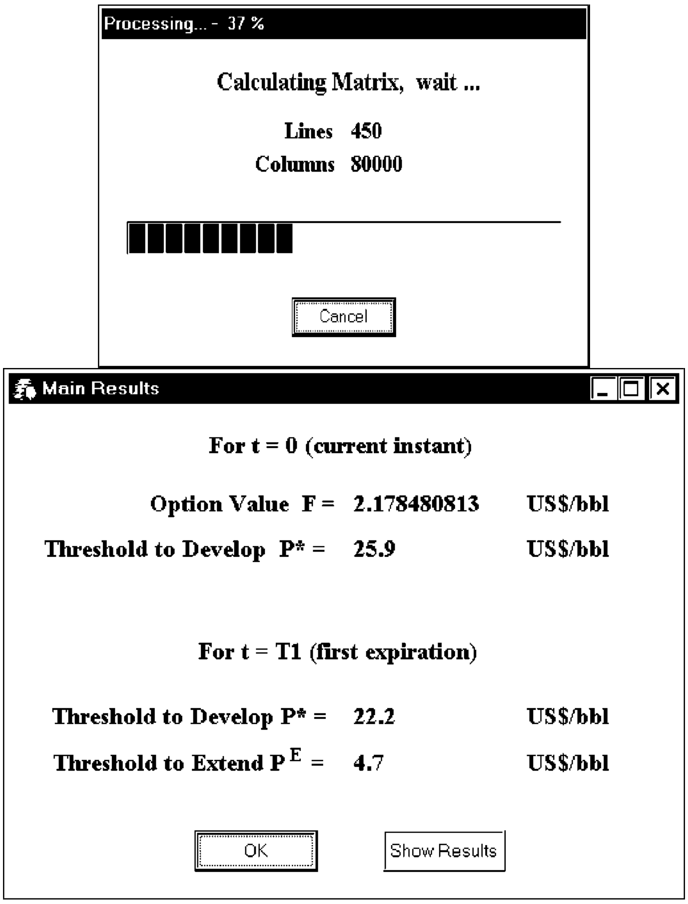
Appendix D
Some Screens from the Software Interface

The software interface was built using Borland C++ Builder. The main screen is shown in the first figure bellow. The interface have three stochastic processes available to choose to perform the calculus for the extendible option problem: *a*) mean-reversion+jump, with the random jump using two truncated-normal distribution (2 Normals); *b*) mean-reversion+jump, with the random jump using log-normal distribution[31] (LogNormal); and *c*) geometric Brownian motion. The parameters from the base case for the first stochastic process are shown in the figure (including the grid density parameters for the finite difference method: ΔP, Δt and P maximum).

The other two figures show the progress of calculations (in order to have an idea of time to complete the calculus) and the main results screen, which has a button to access the complete table of results. There are others screen that are not shown ("load parameters", "help", and so on).



[31] Our first version of the model, presented in Stavanger (May 1998), used the log-normal distribution for jumps [like Merton (1976)] instead the two-truncated normal distribution, so it remained in the software.


Processing... - 37 %
Calculating Matrix, wait ...
Lines 450
Columns 80000
Cancel
Main Results
For t = 0 (current instant)
Option Value F = 2.178480813 US$/bbl
Threshold to Develop P* = 25.9 US$/bbl
For t = T1 (first expiration)
Threshold to Develop P* = 22.2 US$/bbl
Threshold to Extend P E = 4.7 US$/bbl
OK
Show Results

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