MINISTÉRIO EXTRAORDINÁRIO PARA O PLANEJAMENTO E CO-
ORDENAÇÃO ECONÔMICA
ESCRITÓRIO DE PESQUISA ECONÔMICA APLICADA - EPEA

THE RELATION BETWEEN THE RATE OF RETURN ON INVESTMENT PROJECTS AND THE RATE OF GROWTH OF THE NATIONAL PRODUCT.

John S. Chipman

A theoretical background paper, preliminary version
for discussions purposes.

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Rio de Janeiro, August 17, 1965

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I INTRODUCTION

Most post-war analysis of economic growth has proceeded in terms of capital-output ratios, whether with the assumption of fixity in these coefficients (as in the models of Harrod and Domar) or without it (as in those of Solow, Swan, Meade, and Uzawa). This is a fairly useful approach for analyzing growth by means of investment in fixed capital, for it leads to the very simple formula of Harrod (*)

$$(1.1) \qquad g = \frac{\Delta Y}{Y} = \frac{\Delta Y}{\Delta K} \cdot \frac{\Delta K}{Y} = \frac{Y}{K} \cdot \frac{S}{Y} = \frac{s}{k}$$

which states that if there is a fixed capital-output ratio $k = K/Y$ (hence a marginal capital-output ratio equal to the average) and if there is a constant average (hence also marginal) propensity to save $s = S/Y$, and finally if saving is always transformed into net investment ($S=\Delta K$), then the warranted rate of growth g is equal to the propensity to save divided by the capital-output ratio. For example, if the capital-output ratio is $k = 3\frac{1}{3}$ and the propensity to save is $s = 0.2$, then the warranted rate of growth is $g = .06$ or six per cent.

(*) - R F Harrod : "Towards a Dynamic Economics"

This formula can be stated somewhat differently also. If an investment is undertaken and the resulting capital equipment is perfectly durable (or else maintained intact by replacement), then the consequence of an investment of ΔK in year 0 is a perpetual sequence of returns ΔY, ΔY, ... in succeeding years. The reciprocal of the capital-output ratio is therefore simply the rate of return $r = \Delta Y / \Delta K = 1/k$, or "average social rate of return" in Domar's terminology (*). Harrod's formula can therefore be restated as

$$(1.2) \qquad g = rs.$$

In the above example, a capital-output ratio of $3\frac{1}{3}$ would correspond to a rate of return of 30%. It is the purpose of the present paper to show that formula (1.2) holds quite generally; moreover it will be shown that there exist certain natural conditions according to which, in conformity to Solow's results as opposed to those of Harrod, the growth path will be stable, in the sense that the growth process will tend to converge towards it.

This revised formulation has the adventage of making possible a comparison of different types of investment projects, including such projects as investment in education which cannot be analyzed very successfully in terms of capital-output ratios. The result that we obtain also gives strong support to the rate-of-return approach in evaluating alternative investment projects.

II. THE RATE OF RETURN

I shall confine myself at first to consideration of investment projects characterized by a single initial investment of one unit followed by a finite sequence of positive returns a_i in the ensuing years:

(*) - Evsey Domar : Essays on Capital and Growth

$$(2.1) \qquad -1,\ a_1,\ a_2,\ \ldots,\ a_n$$

Later I shall consider more complicated cases, but it will be instructive to analyze this case first. The rate of return is defined - following Fisher (*) - as the largest interest rate r_1 for which the present value of this sequence of costs and returns vanishes:

$$(2.2) \qquad V\ (1+r) = -1 + \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + \ldots + \frac{a_n}{(1+r)^n} = 0$$

This is equivalent to saying that the return factor $1+r_1$ is the largest root of the polynomial

$$(2.3) \qquad f(1+r) = (1+r)^n - a_1(1+r)^{n-1} - a_2(1+r)^{n-2} - \ldots - a_n$$

One obvious reason for choosing the largest of the n roots of this polynomial is that, since obviously $V(1+r)$ is negative for sufficiently large r, it follows by continuity that it is negative for all r greater than r_1, and positive for all r in the interval $r_2 < r < r_1$, where $1+r_2$ is the next largest real root of the polynomial $f(1+r)$. Now since the sequence (2.1) contains only one change of sign, it follows by Descartes' Rule of Signs (**) that $f(1+r)$ has exactly one positive root. Now only the positive roots $1+r$ generally have any economic meaning, for a lender cannot lose more than the total amount lent, and the case in which the full amount lent was lost would correspond to a zero interest factor $1+r = 0$, that is a rate

(*) Irving Fisher : The Theory of Interest, and The Nature of Capital and Income. Fisher somewhat carelessly defined the internal rate of return (or "rate of return over cost" in his terminology) as being "that interest rate" for which the present value of the sequence of yields would be equal to the initial cost. The same was done by J.M. Keynes in the General Theory when defining the marginal efficiency of capital. The reason for choosing the largest interest rate will become obvious presently.

(**) Cf., for example, Louis Weisner : Introduction to the Theory of Equations. New York: The MacMillan Co., 1938, pp. 88-9 of interest of -100%. This remains true of the real rate of interest during a period of inflation, since if i is the market rate of interest and p the rate of inflation, then one dollar invested now will become $1+i$ dollars in the following year, but this quantity will have suffered a fall in purchasing power of $100p\%$ and consequently be worth only

$$1 + r = \frac{1 + i}{1 + p} \approx 1 + i - p$$

in constant dollars. As the rate of inflation approaches infinity, $1 + r$ approaches zero and r aproaches the minimum possible value of -1. Thus for all conceivable real rates of interest r, the project represented by (2.1) will be profitable whenever the interest rate is less than the rate of return, and unprofitable otherwise.

The above discussion has bearing on the question of how an efficient allocation of saving can be achieved by means of a capital market, provided the criterion of evaluating a project by the rate of return can be justified on independent grounds. It is the latter question which is the object of interest in the discussion to follow.

III. THE GROWTH PROCESS

I shall assume that the gross national product in year t, P_t, can be divided into two parts: an autonomous part Z which is assumed to be constant, and an induced part which consists of the returns from previous gross investments G_{t-i} carried out i years earlier, $i = 1, 2, \ldots, n$. The returns a_i of (2.1) will be interpreted as being net of any maintenance expenses, but excluding depreciation on fixed capital that must be replaced. Assuming that a_i is the constant proportion of the return in any year t to the gross investment

G_{t-i} i years previously, the gross national product in year t will be

$$(3.1) \quad P_t = Z + a_1 G_{t-1} + a_2 G_{t-2} + \dots + a_n G_{t-n}$$

Similarly, let us assume than the depreciation of a gross capital investment if one unit is spread out during the subsequent n years of life of the capital according to the sequence

$$(3.2) \quad -1,\ d_1,\ d_2,\ \dots,\ d_n$$

where

$$(3.3) \quad d_1 + d_2 + \dots + d_n = 1.$$

Then in any year t the depreciation expense D_t of that year will be a snapshot consisting of the depreciation expenses $d_i\ G_{t-i}$, assumed to be a constant proportion of the previous gross investments:

$$(3.4) \quad D_t = d_1\ G_{t-1} + d_2\ G_{t-2} + \dots + d_n\ G_{t-n}$$

Net national product in year t is therefore given by

$$(3.5) \quad Y_t = P_t - D_t = Z + (a_1 - d_1) G_{t-1} + (a_2 - d_2)\ G_{t-2} + \dots + (a_n - d_n) G_{t-n}$$

Now let net investment I_t be a fixed proportion s of the current net national product Y_t, where s is the fixed average propensity to save:

$$(3.6) \quad I_t = sY_t.$$

Gross investment is then

$$(3.7) \quad G_t = I_t + D_t = sY_t + d_1 G_{t-1} + d_2 G_{t-2} + \dots + d_n G_{t-n}$$

From (3.5) we may replace t successively by t-1, t-2,...,t-n and multiply through respectively by $-d_1$, $-d_2$, ..., $-d_n$, to obtain the equations

$$Y_t = Z + (a_1 - d_1)\ G_{t-1} + (a_2 - d_2)\ G_{t-2} + \dots + (a_n - d_n)\ G_{t-n}$$

$$-d_1 Y_{t-1} = -d_1 Z - (a_1 - d_1) d_1 G_{t-2} - (a_2 - d_2) d_1 G_{t-3} - \dots - (a_n - d_n) d_1 G_{t-n-1}$$

$$-d_2 Y_{t-2} = -d_2 Z - (a_1 - d_1) d_2 G_{t-3} - (a_2 - d_2) d_2 G_{t-4} - \dots - (a_n - d_n) d_2 G_{t-n-2}$$

$$\vdots$$

...
...
...
...

$-d_nY_{t-n}=-d_nZ-(a_1-d_1)d_nG_{t-n-1}-(a_2-d_2)d_nG_{t-n-2}-\dots-(a_n-d_n)d_nG_{t-2n}$

Adding up both sides and making use of (3.3), and observing from (3.7) that

$$G_t - d_1 G_{t-1} - d_2 G_{t-2} - \dots - d_n G_{t-n} = sY_t ,$$

we obtain

$$(3.8) \quad Y_t - d_1Y_{t-1} - d_2Y_{t-2} - \dots - d_nY_{t-n} = (a_1-d_1)sY_{t-1} + (a_2-d_2)sY_{t-2} + \dots + (a_n-d_n)sY_{t-n}$$

whereupon the fundamental difference equation is

$$(3.9) \quad Y_t - [d_1+(a_1-d_1)s]\, Y_{t-1} - [d_2+(a_2-d_2)s]\, Y_{t-2} - \dots - [d_n+(a_n-d_n)s] = 0$$

which may also be written in the form

$$(3.10) \quad Y_t - [sa_1+(1-s)d_1]\, Y_{t-1} - [sa_2+(1-s)d_2]\, Y_{t-2} - \dots - [sa_n+(1-s)d_n] = 0$$

expressing the coefficients as weighted averages of the returns and depreciation expenses, the weights being the propensity to save and the propensity to consume.

Since equation (3.10) is homogeneous, it it known to have a solution of the form

$$(3.11) \quad Y_t = A_1x_1^{\,t} + A_2x_2^{\,t} + \dots + A_nx_n^{\,t}$$

where the x_i are the roots of the characteristic polynomial

$$(3.12) \quad g(x)=x^n-[sa_1+(1-s)d_1]\, x^{n-1} - [sa_2-(1+s)d_2]\, x^{n-2} - \dots - [sa_n + (1-s)\, d_n] ,$$

and where the constants A_i are determined by the initial conditions.

The problem as formulated so far is incomplete, since no rule has yet been specified for the allowance for depreciation. Such a rule will now be adopted in the light of the following theorem.

Theorem 1. The polynomial (3.12) has $x = 1+r_1 s$ as one of its roots - where $1+r_1$ is the largest real root of the polynomial (2.3) - if and only if, for arbitrary s,

$$\begin{aligned} d_1 &= a_1 - r_1 \\ d_2 &= a_2 - r_1 (1-d_1) \\ d_3 &= a_3 - r_1 (1-d_1 - d_2) \\ &\cdots \\ &\cdots \\ &\cdots \\ d_n &= a_n - r_1 (1-d_1 - d_2 - \dots - d_{n-1}) \end{aligned}$$

The proof is straight forward but tedious, and will therefore be omitted. It is also readily verified that (3.3) holds, use being made of the fact that $1 + r_1$ is a root of (2.3). The depreciation rule has a very natural interpretation: in the first year, enough depreciation is deducted to make the net return a_1-d_1 equal to the original investment of 1 unit multiplied by the internal rate of return r_1 ; since the original capital has been valued down to $1-d_1$ units, in the second period the depreciation d_2 will be such as to make the net return a_2-d_2 equal to the revalued capital $1-d_1$ multiplied by the rate of return r_1 ; and so on.

As a corollary to Theorem 1 it is readily verified that the polynomial (3.12) factors as follows:

$$(3.13) \quad g(x) = [x-(1+r_1 s)] \, [x^{n-1}+(1-d_1)x^{n-2}+(1-d_1-d_2)x^{n-3}+\dots \\ \dots+(1-d_1-d_2-\dots-d_{n-1})]$$

Putting $x_1 = 1+r_1 s$, in order for the solution (3.11) to converge to the growth path $A_1 (1+r_1 s)^t$ for arbitrary initial conditions, we require that the roots $x_2, x_3, \dots, x_n$ of

$$(3.14) \quad h(x)=x^{n-1}+(1-d_1)x^{n-2}+(1-d_1-d_2)x^{n-3}+\dots+(1-d_1-d_2-\dots \\ \dots-d_{n-1})$$

be less than unity in absolute value. Before taking up this problem, let us analyze a simple numerical example in which

$$A_1 = Y_0 \text{ and } A_2 = A_3 = \ldots = A_n = 0.$$

IV. A NUMERICAL EXAMPLE

Let us consider s simple case of a project characterized by a single investment of one unit followed by two successive returns:

(4.1) $(-1, a_1, a_2) = (-1, .7, .78)$.

We verify that the polynomial (2.3) becomes

$$f(1+r)=(1+r)^2-.7(1+r)-.78=(1+r-1.3)(1+r+.6),$$

whence the two roots of f are

$$1 + r_1 = 1.3 \;; \quad 1 + r_2 = - .6.$$

Thus the rate of return is 30%. Accordingly we may calculate the appropriate depreciation expenses from the rule given by Theorem 1 :

$$d_1 = a_1 - r_1 = .7 - .3 = .4$$

$$d_2 = a_2 - r_1 (1-d_1) = .78 - .3(.6) = .6$$

Thus we have

(4.2) $(-1, d_1, d_2) = (-1, .4, .6)$

and obviously (3.3) is satisfied. Assuming $s = .2$, we obtain from (3.12) and (3.13) the polynomial

$$g(x) = x^2 - .46x - .636 = (x-1.06)(x+.6)$$

whose roots are therefore

$$x_1 = 1+r_1 s = 1.06; \; x_2 = d_1 - 1 = -.6.$$

Thus the equilibrium rate of growth is six per cent, and it is stable since $x_2 = -.6$ which is less than unity in absolute value.

Let us assume the process to begin with gross and net national product of 1000. The sequence of events is traced out in the following table:

TABLE 1

Year	0	1	2	3	4
P	1 000	1 140	1 360.40	1 550.8240	
D	-	80	236.80	359.8080	
Y	1 000	1 060	1 123.60	1 191.0160	
I	200	212	224.72	238.2032	
C	800	848	898.88	952.8128	
G	200	292	461.52	598.0112	
	- 200	140	156.00		
		- 292	204.40	227.7600	
			- 461.52	323.0640	359.9856
	- 200	80	120.00		
		- 292	116.80	175.2000	
			- 461.52	184.6080	276.9120

As a consequence of the investment of 200 in year 0 there are, according to (4.1), returns of 140 and 156 in the two succeding years, as shown below the first part of the table; also deprection of 80 and 120 as shown in the bottom part. Thus the GNP is 1140 in year 1, whence the NNP is first calculated, then net investment from the formula $I_t = sY_t$, then consumption from the formula $C_t = Y_t - I_t$, and finally the new gross investment from $G_t = I_t + D_t$. The rest of the table should be self-explanatory.

It will be observed that the rate of growth of the national income is 6% each year. This is a consequence of the initial conditions, for in order for the table to be consistent with the assumptions, national income had to be growing at the rate of six per cent during the previous periods; thus we could just as well compute the process backwards as forwards. The "initial conditions" can be expressed in the form

$$A_1 + A_2 = Y_0$$

$$A_1x_1 + A_2x_2 = Y_0\ (1+r_1s). = Y_0x_1.$$

Solving for A_1 and A_2 by multiplying the first equation by x_1 and subtracting the result from the second equation, we obtain

$$A_2\ (\ x_2 - x_1\ ) = 0,$$

and since $x_1 > 0$ and $x_2 < 0$ certainly $x_2 - x_1 < 0$ so $A_2 = 0$, hence $A_1 = Y_0$.

This example might lead one to believe that it is not necessary that the roots of g(x) be less than unity in absolute value, as long as initial conditions can be chosen so that national product stays on the growth path. Quite apart from the difficulty of achieving the latter condition in practice, upon consideration it will become evident that there is another reason for insisting that g(x) have roots within the unit circle in order that the solution be economically meaningful: the reason is that gross investment - according to the model - will eventually become negative if the process is unstable, and obviously this is meaningless. Let us therefore explicity state the condition

$$(4.3) \qquad G_t \geq 0$$

Now in order to see that (4.3) will generally be violated if g(x) has roots greater than unity in absolute value, we observe from (3.7) and (3.8) that

$$(4.4) \qquad G_t - [sa_1+(1-s)d_1]\ G_{t-1} - [sa_2+(1-s)d_2]\ G_{t-2} - \cdots$$
$$\cdots - [sa_n+(1-s)d_n]\ G_{t-n} = sZ.$$

This difference equation has the same form as (3.10) except than it is no longer homogeneous. The equilibrium solution, defined by

$$G' = G_t = G_{t-1} = \dots = G_{t-n}$$

is

$$(4.6) \qquad G' = \frac{Z}{1-a_1-a_2-\dots-a_n}$$

which is negative (since with a positive rate of return and positive a_i, s, the denominator will be negative). The homogeneous solution, defined by

$$(4.7) \qquad G'' = G_t - G'$$

is given by

$$(4.8) \quad G_t'' = B_1x_1^t + B_2x_2^t + \dots + B_nx^t_n$$

Consequently the general solution of (4.4) is obtained by adding (4.6) and (4.8) to get

$$(4.9) \quad G_t = \frac{Z}{1-a_1-a_2-\dots-a_n} + B_1x_1^t + B_2x_2^t + \dots + B_nx_n^t$$

In order that coefficients B_2, B_3, ... B_n be annihilated it is necessary and sufficient that during the initial n periods the relations

$$(4.10) \quad G_t-Z/(1-a_1-a_2-\dots-a_n) = x \quad G_{t-1}-Z/(1-a_1-a_2-\dots-a_n)$$

should hold. There does not seem to be any particular reason for expecting this result; in particular it does not hold in the example of Table 1.

In the following section I therefore take up the question of the stability of the solution

$$Y_t = (1+r_1s)^tY_0$$

to (3.11), i.e., the conditions under which the polynomial (3.14) has roots all less than unity in absolute value. It is clear that if this condition does not hold then some of the roots x_2, x_3, ..., x_n of (3.13) will be negative or complex with absolute value greater than one, and from (4.9) it is obvious that gross investiment would be eventually become negative, in violation of (4.3).

V. STABILITY OF THE GROWTH PATH

Consider the polynomial

(5.1) $h(x;n) = x^n+(1-d_1)x^{n-1}+(1-d_1-d_2)x^{n-2}+\ldots$

$\ldots+(1-d_1-d_2-\ldots-d_n)$

where it is assumed that

(5.2) $1>1-d_1>1-d_1-d_2>\ldots>1-d_1-d_2-\ldots-d_n>0.$

It is desired to prove that the roots of (5.1) all have absolute value less than 1. Let us state formally:

Theorem 2. The polynomial (5.1) has all its roots within the unit circle provided (5.2) holds.

The theorem is obviously true for n=1, in which case the single root is

$$x_1 = -(1-d_1)$$

which is between 0 and -1. We note also that (5.1) obviously has no positive roots, since $h(x;n)$ is positive for all positive values of x. Its roots must therefore be either negative or complex.

The truth of theorem 2 for $n = 2$ is readily established by means of direct solution by radicals. In that case the discriminant is

(5.3) $D=(1-d_1)^2-4(1-d_1-d_2)=(1+d_1)^2-4(1-d_2)$

Suppose first that $D \geq 0$, whence the roots are real. Since both are negative, we require that the smaller of the two (say x_2) be greater than -1, or equivalently

$$2x_2 = -(1-d_1)-\sqrt{D} > -2$$

which leads to the condition $(1-d_1)^2 > D$ which follows from (5.3) if and only if $d_2 < 1$ as assumed.

Now suppose that $D < 0$; then the two roots may be written $x_1 = u + vi$ and $x_2 = u - vi$, and we require that $u^2 + v^2 < 1$, or equivalently,

$$(1-d_1)^2 + D < 4$$

and this reduces to

$$d_1^2 + 2d_2 < 3$$

which certainly follows, since both d_1 and d_2 are between zero and one.

As a prelude to taking up the general case, consider the first derivative of the polynomial (5.1):

$$(5.4) \quad h'(x;n)=nx^{n-1}+(n-1)(1-d_1)x^{n-2}+(n-2)(1-d_1-d_2)x^{n-3}+\dots$$
$$\dots+2(1-d_1-d_2-\dots-d_{n-2})x+(1-d_1-d_2-\dots-d_{n-1})$$

First we prove:

<u>Lemma 1</u>. For all $x \leq -1$, $h'(h;n)$ is positive if n is odd and negative if n is even.

<u>Proof</u>. If n is odd, there are an odd number (n) of terms on the right side of (5.4), the last of which is positive by assumption (5.2). We shall show that each successive pair of terms is positive, i.e., for i even,

$$(5.5) \quad (n-i)(1-d_1-d_2-\dots-d_i)x^{n-i-1}+(n-i-1)(1-d_1-d_2-\dots$$
$$\dots-d_i-d_{i+1})x^{n-i-2} > 0.$$

Clearly $n-i > n-i-1 > 0$, and using (5.2) we observe that the first coefficient of (5.5) is larger than the second, and that x^{n-i-1} (which is positive since $n-i-1$ is even) is not smaller in absolute value than x^{n-i-2}; thus (5.5) follows.

Now if n is even we follow the same procedure, considering all pairs such as (5.5) where i is even, and which exhaust the right side of (5.4). This time x^{n-i-1} will be negative since $n-i-1$ is odd, and will have absolute value greater than or equal to x^{n-i-2}, hence the inequality sign in (5.5) will be reversed and the lemma follows.

<u>Lemma 2</u>. Given assumption (5.2), the polynomial (5.1) has no real roots less than -1.

<u>Proof</u>. Observe that (5.1) may be written in the recursive form.

(5.6) $h(x;n) = xh(x;n-1)+(1-d_1-d_2-\ldots-d_n)$

Suppose n is odd; then from (5.1) we verify that

(5.7) $h(-1;n-1) = 1-d_2-d_4-\ldots-d_{n-1} > 0$ (n odd).

Since $n-1$ is even, it follows from Lemma 1 that for all $x \leqq -1$, $h'(x;n-1) \leqq 0$; therefore

(5.8) $h(x;n-1) \geqq 1-d_2-d_4-\ldots-d_{n-1} > 0$ for $x \leqq -1$.

Consequently, if x_n were a root of (5.6) and $x_n \leqq -1$ we would have from (5.6) and (5.8)

(5.9a) $x_n h(x_n;n-1) + 1-d_1-d_2-\ldots-d_{n-1}-d_n = 0$

(5.9b) $x_n h(x_n;n-1)+1-d_2-d_4-\ldots d_{n-1} \leqq 0$

where (5.9b) is obtained by multiplying (5.8) through by the negative quantity $x_n \leqq -1$. Subtracting (5.9b) from (5.9a) we obtain:

$$-d_1-d_3-\ldots-d_n \leqq 0$$

in contradiction to (5.2). Therefore $x_n \leqq -1$ could not have been a root of (5.1).

Now suppose n is even; then from (5.1) we verify that

(5.10) $h(-1;n-1) = -d_1-d_3-\ldots-d_{n-1} < 0$ (n even).

Since $n-1$ is odd, it follows from Lemma 1 that for all $x \leqq 1$, $h'(x;n-1) > 0$, therefore

(5.11) $h(x;n-1) \leqq -d_1-d_3-\ldots-d_{n-1} < 0$ for $x \leqq -1$.

Thus if x_n were a root of (5.6) and $x_n \leqq -1$ we would have from (5.6) and (5.11)

(5.12a) $x_n h(x_n;n-1)+1-d_1-d_2-\ldots-d_{n-1}-d_n=0$

(5.12b) $x_n h(x_n;n-1) \quad -d_1-d_3-\ldots-d_{n-1} \geqq 0$

where (5.12b) is obtained by multiplying (5.11) through by $x_n \leqq -1$. Subtracting (5.12b) from (5.12a) we obtain

(5.13) $1-d_2-d_4-\ldots-d_n \leqq 0$

which is again in contradiction to (5.2).

Q.E.D.

In order to complete the proof of Theorem 2, it is still necessary to show that the complex roots of (5.1) all have modulus less than unity. The methods used up to this point do not appear to be adequate to provide a solution to this problem, and a different approach will be required. Let us then avail ourselves of conditions established by Schur.*

It will be convenient to define

$$(5.14) \qquad c_i = i-d_1-d_2-\dots-d_{n-i} \qquad (i=0,1,2,\dots,n-1)$$

where we define $c_n=1$. Accordingly the polynomial (5.1) may be written

$$(5.15) \qquad h(x;n) = c_0+c_1x+c_2x^2+\dots+c_nx^n$$

where, corresponding to conditions (5.2), we have

$$(5.16) \qquad 0 < c_0 < c_1 < c_2 < \dots < c_n = 1.$$

Now it was established by Schur ** that the polynomial (5.15) has all its roots within the unit circle (whether or not (5.16) holds) if and only if the n determinants

$$(5.17) \qquad \Delta_i = \begin{vmatrix} C_{n,i} & C_i' \\ C_i & C_{n,i}' \end{vmatrix} \qquad (i = 0,1,2,\dots,n-1)$$

are all positive, where we define the matrices

$$(5.18) \qquad C_{n,i} = \begin{bmatrix} c_n & 0 & 0 & \dots & 0 \\ c_{n-1} & c_n & 0 & \dots & 0 \\ c_{n-2} & c_{n-1} & c_n & & 0 \\ \vdots & \dots & \dots & \dots & 0 \\ c_{n-i} & c_{n-i+1} & c_{n-i+2} & \dots & c_n \end{bmatrix}$$

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* J. SCHUR, " Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind ", Journal für die reine und angewandte Mathematik (gegründet von A.L. Crelle 1826) Band 147 (1917), 205-232, Band 148 (1918), 122-145.

** Op. cit., pp. 134-5. Schur's formula is more general, allowing for the possibility that the coefficients c_i of (5.15) be complex. This possibility obviously does not concern us here and

$$(5.14)\ C_i = \begin{bmatrix} c_0 & 0 & 0 & \dots & 0 \\ c_1 & c_0 & 0 & \dots & 0 \\ c_2 & c_1 & c_0 & \dots & 0 \\ & & & & \\ c_i & c_{i-1} & c_{i-2} & \dots & c_0 \end{bmatrix}$$

Schur also showed* that if the matrices C_i and $C_{n,i}$ commute, i.e., if $C_i C_{n,i} = C_{n,i} C_i$, then

$$(5.20)\quad \Delta_i = \left| C_{n,i} C'_{n,i} - C_i C'_i \right| \qquad (i = 0,1,2\ldots,n-1)$$

and it is readily verified that this commutativity condition holds. Thus in order to establish Theorem 2 it is sufficient to verify from either (5.17) or (5.20) that the determinants Δ_i are positive given that condition (5.16) holds.

* Op. cit., pp. 216-17. This formula holds for any partitioned matrix satisfying the commutativity condition.